§5.4 Properties of Estimators

In §5.2, we use the maximum likelihood method to estimate parameters θ (or λ) in the **pdf** $p_X(x;\theta)$ of a random variable X, based on a sample (observations) $X_1 = k_1$, $X_2 = k_2$, ... , $X_n = k_n$.

Maximum likelihood gives one method for estimation θ_e. There are some other methods to estimate θ. We want to know which estimate is better. We study the **estimator** $\widehat{\theta}$ which is a function of the random variables X_1, X_2, ... , X_n for each estimate θ_e.

Example 1. (Example 3 in §5.2.) The random variable X fits a Poisson distribution

$$p_X(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}, \text{ for } k = 0, 1, 2, ...$$

Based on 4 observations $X_1 = k_1$, $X_2 = k_2$, $X_3 = k_3$, $X_4 = k_4$, the maximum likelihood estimate (MLE) is

$$\lambda_e = \frac{k_1 + k_2 + k_3 + k_4}{4}$$

The corresponding **estimator** is

$$\widehat{\lambda} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

The estimator $\widehat{\theta}$ itself is a random variable. We can study its mean $E(\widehat{\theta})$ and variance $\text{Var}(\widehat{\theta})$.

Two important properties of estimators (unbiasedness and efficiency) are defined by mean and variance. We focus on **unbiasedness** first.

Definition.

An estimator $\widehat{\theta}$ is called **unbiased** if $E(\widehat{\theta}) = \theta$.

In Example 1,

$$\begin{aligned}E(\widehat{\lambda}) =& E(\frac{X_1 + X_2 + X_3 + X_4}{4}) \\ =& \frac{E(X_1) + E(X_2) + E(X_3) + E(X_4)}{4} = \frac{4\lambda}{4} = \lambda\end{aligned}$$

So, the estimator λ is unbiased.

Example 2. A sample of size two includes Y_1 and Y_2 from the same **pdf** . Suppose we have a estimator for the mean μ defined by

$$\widehat{\mu} = cY_1 + (1 - c)Y_2, \quad 0 \leq c \leq 1.$$

(1) For which c, the above statistic is an unbiased estimator for μ?

(2) Is $\widehat{\mu} = 0.2Y_1 + 0.6Y_2$ an unbiased estimator for μ?

Example 3. Suppose $X \sim Binomial(n, p)$. That is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Find the maximum likelihood estimate(MLE) for p based on a sample $X_1 = x_1$, $X_2 = x_2$, ... , $X_m = x_m$. Is the corresponding estimator unbiased?

Example 4. Suppose $X \sim \text{Normal}(\mu, \sigma^2)$. That is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty.$$

Find the maximum likelihood estimate (MLE) for μ and σ based on a sample $X_1 = x_1$, $X_2 = x_2$, ... , $X_n = x_n$. Are the corresponding estimators unbiased?

Definition.

Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be two unbiased estimators. We say that $\widehat{\theta}_1$ is more **efficient** than $\widehat{\theta}_2$ if $\text{Var}(\widehat{\theta}_1) < \text{Var}(\widehat{\theta}_2)$.

Example 5. Let Y_1, Y_2 and Y_n be random sample of size n from the pdf $f_Y(y;\theta) = \frac{1}{\theta}e^{-y/\theta}$, $y \geq 0$.

(1) Show that both $\hat{\theta}_1 = Y_1$ and $\hat{\theta}_2 = \overline{Y}$ are unbiased estimator for θ.

(2) Find the variance of $\hat{\theta}_1 = Y_1$ and $\hat{\theta}_2 = \overline{Y}$.