§4.2 Poisson Distribution

Definition. (Poisson Distribution)

The **Poisson Distribution** Piosson(λ) is a discrete **pdf** function defined as

$$p_X(k) = P(X = k) := \frac{\lambda^k e^{-\lambda}}{k!}$$

for $k = 0, 1, 2, 3, ...$ Here, λ is a positive constant.

Theorem.

(1) It is a well defined **pdf** , i.e., $\sum_k p_X(k) = 1$
(2) The mean is $E(X) = \lambda$.
(3) The variance is $\text{Var}(X) = \lambda$.

Proof:
(1)

$$\sum_{\text{all } k} p_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

(2)

$$\begin{aligned} E(X) &= \sum_{\text{all } k} k p_X(k) = \sum_{k=0}^{\infty} \frac{k \lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} = \lambda e^{-\lambda} e^{\lambda} = \lambda \end{aligned}$$

(3) Similarly as (2) but more tricky.

Poisson Distribution Piosson(λ):

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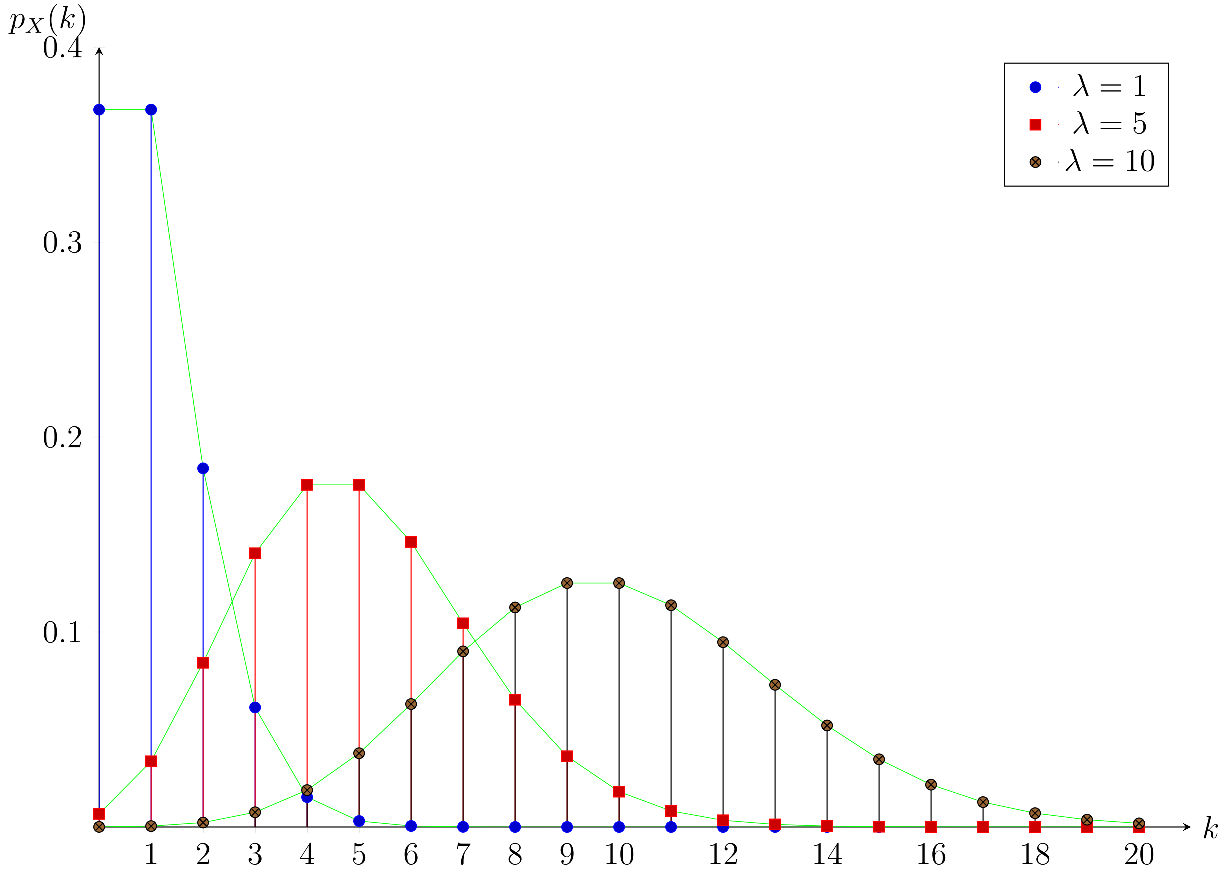
for $k = 0, 1, 2, 3, ...$



Poisson distribution is discrete. The graphs only contain the discrete points. Green lines are not part of the graph.

Historically, Poisson distribution is used as an approximation for binomial distribution

$$p_Y(k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}, \text{ for } k = 0, 1, 2, \ldots, n.$$

Applications of Poisson distribution:

1. Poisson approximation for binomial distribution.

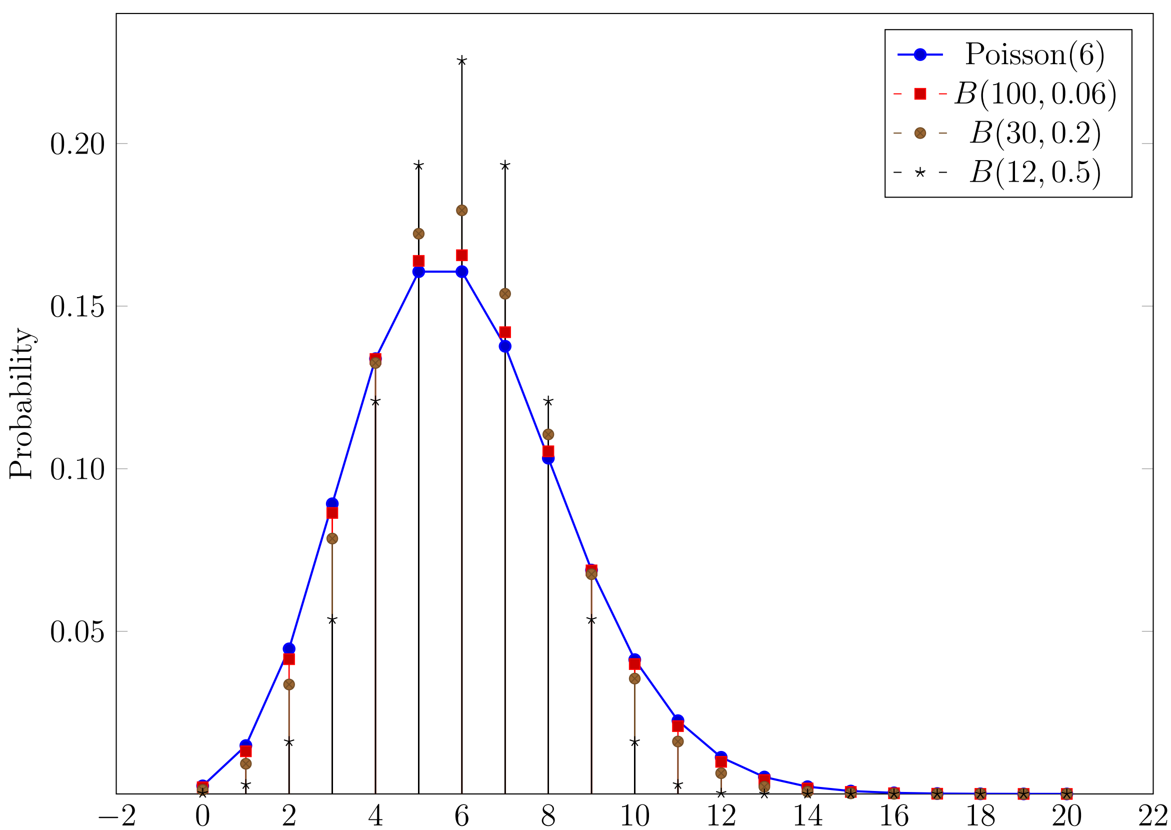
Theorem. Poisson limit

If n is large and p is small, then let $\lambda = np$, we have

$$\frac{\lambda^k e^{-\lambda}}{k!} \approx \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

More precisely, if $np = \lambda$ is constant,

$$\lim_{\substack{n\to\infty \\ p\to 0}} \binom{n}{k} \cdot p^k (1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}$$



Example 1. Airline company knows that on average 5% of people will *not show up* on a flight. There are 98 seats, and the airline sells 100 tickets. Use the Poisson approximation to find the probability that everyone gets a seat.

Solution:

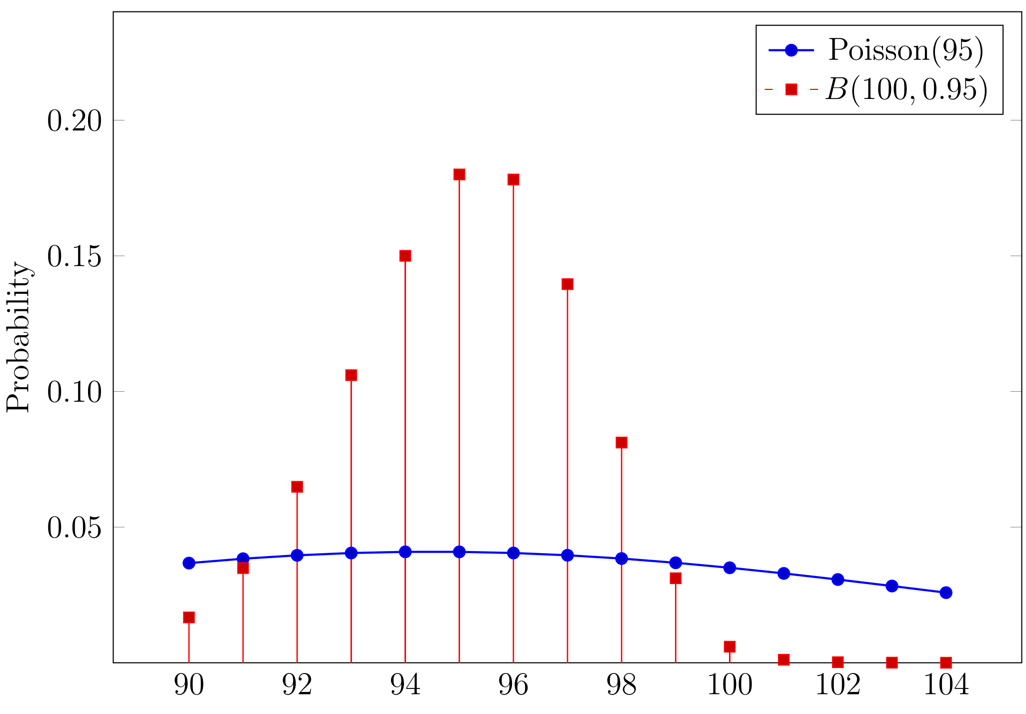
Y : Number of people *not show up*.

Y admits the binomial distribution (3 conditions: independent, fixed probability, two outcomes). $n = 100$ and $p = 0.05$. **Large** n **small** p. We can use the Poisson approximation with $\lambda = np = 5$.

"Everyone gets a seat" means that "≥ 2 people not show up".
So,

$$\begin{aligned} P(Y \geq 2) &= 1 - P(Y = 0) - P(Y = 1) \\ &\approx 1 - \frac{5^0 e^{-5}}{0!} - \frac{5^1 e^{-5}}{1!} \\ &\approx 0.96 \end{aligned}$$

If we let X be the number of people **show up**, we can use the binomial distribution to calculate. But since $p = 0.95$ which is **not** small, we can **not** use Poisson to do the estimation.



Example 2. (Practice) Suppose there is a rare disease. One in a million people have this disease. In a city with 3 million people, find the probability that more than 2 people have this disease.

Solution:

X : Number of people in this city has the disease.

X admits the binomial distribution (3 conditions!). $n = 3\ million$ and $p = 1/million$.
Large n **small** p. We can use the Poisson approximation with $\lambda = np = 3$.
So,

$$\begin{aligned} P(X > 2) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &\approx 1 - \frac{3^0 e^{-3}}{0!} - \frac{3^1 e^{-3}}{1!} - \frac{3^2 e^{-3}}{2!} \\ &\approx 0.5768 \end{aligned}$$

Example 3. (Practice) Suppose the probability that a pixel in a smart phone is defective is 1 in a million. If your phone has 3 million pixels, find the probability that more than 2 pixels are defective in your phone.

The same as Example 2

Example 4. (5 points) Suppose that your router handles 1 million packets a second. For each packet it handles assume that there is an (independent) one in five million chance that it will be dropped. What are the chances that your router drops exactly one packet in the next second? (Use an appropriate approximation)

Answer: By Poisson approximation with $\lambda = np = 1million(\frac{1}{5million}) = 0.2$,

$$e^{-0.2}\frac{0.2}{1!} \approx 0.16$$

2. Poisson Model. In natural, there are data fit Poisson distribution.

Poisson Model: Some events happen **independently** in a **time interval** T. The probability for each event occur is **constant**. We can break the time interval T to small part T/n to make sure that no two events occur in the same small part.

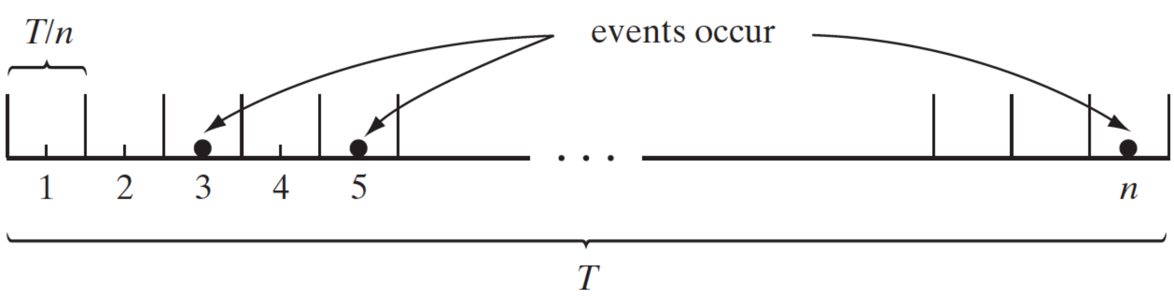
Let X be the number of events that occur in a time interval. Then X admit the Poisson distribution for some number λ,

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, 3, ...$$

Poisson model: The number of (...) occurs with a rate of (...)/**time**.

1. The number of people come to (bank/store) with a **rate** of (...) per **hour**.
2. The number of phone calls/messages with an **average** of (...) per **hour**.
3. The number of (traffic/job/system) accidents/errors with an **rate** of (...) per **day**.
4. The weight of bug parts a person eats with a **average** of (...) pound per **year**.
5. The number of of rain drops in one minute.
6. The number of of cars passing a traffic light for one hour.
7. The number of of typos/a word in one 1000 words.
9. The number of of visitors of a certain web site in one hour.

...



Find λ is the **key** to use Poisson model.
$\lambda = E(X)$=The expected value (mean) of **X**.

Example 5. The number of people who go to a supermarket is 10 people per 2 minutes.

(1) Find the probability that exactly 2 people come into the supermarket in the next **10 seconds**.

Solution: Let Z be the number of people come in the next **10 seconds**. So, $\lambda = E(Z) = 10(10/120) = 5/6$.

$$P(Z=2) = \frac{(5/6)^2 e^{-5/6}}{2!} \approx 0.151$$

(2) Find the probability that exactly 3 people come into the supermarket in the next **minute**.

Solution: Let Y be the number of people come in the next **minute**. So $\lambda = E(Y) = 10/2 = 5$.

$$P(Y=3) = \frac{5^3 e^{-5}}{3!} \approx 0.1403738$$

(3) Use a different way to calculate (2). Let X be the number of people come in the next **half minute**.

Solution: Since we let X be the number of people come in the next half minute, so $\lambda = E(X) = \frac{1}{2}(10/2) = 2.5$.

$$\begin{aligned} &P(Y=3) \\ &= P(X=3)P(X=0) + P(X=2)P(X=1) + P(X=1)P(X=2) + P(X=0)P(X=3) \\ &\approx (0.2138)(0.0821) + (0.2565)(0.2052) + (0.0821)(0.2138) + (0.2052)(0.2565) \\ &\approx 0.1403735 \end{aligned}$$

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For Example, in (2), we can compute

$P(Y=3) =$ **poissonpdf**(5,3)=0.1403738

Remark: If the number of occurrences is **Poisson** distribution with mean λ, then the time between occurrences is the **exponential** distribution

$$f_Y(y) = \lambda e^{-\lambda y}$$

for $y \geq 0$.