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1 Quadratic forms and positive definite

Definition.

A function $p(x_1, ..., x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is call a **quadratic form**, if it is a linear combination of forms $x_i x_j$.

So, a quadratic form can be written as

$$p(x_1, ..., x_n) = \sum_{i,j} c_{ij} x_i x_j = c_{11}x_1^2 + c_{22}x_2^2 + \cdots + c_n x_n^2 + \sum_{i<j} \mathbf{2} c_{ij} x_i x_j$$

Another way to write quadratic form is using symmetric matrices

$$p(x_1, ..., x_n) = \vec{x} \cdot A\vec{x} = \vec{x}^T A \vec{x}$$

The unique symmetric matrix A is called the **matrix for the quadratic form**.

The matrix A for the above quadratic form is $A = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \dots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$.

Example 1. Consider $p(x_1, ..., x_3) = 3x_1^2 + 4x_2^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 6x_2x_3$

The matrix A for this quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & -5 \end{bmatrix}$$

You can check that $p(x_1, ..., x_3) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Definition.

An real symmetric matrix A is called **positive definite** if the quadratic form

$$\vec{x}^T A\vec{x} > 0$$

for all nonzero $\vec{x} \in \mathbb{R}^n$.
The matrix A is called **positive semidefinite** if the quadratic form

$$\vec{x}^T A\vec{x} \geq 0$$

for all $\vec{x} \in \mathbb{R}^n$.

Examples: 1. $p(x_1, ..., x_3) = 2x_1^2 + 3x_2^2 + 4x_3^2$ is positive definite.

2. $p(x_1, ..., x_3) = 2x_1^2 + 3x_2^2$ is positive semidefinite.

3. $p(x_1, ..., x_3) = 2x_1^2 + 3x_2^2 - 4x_3^2$ is NOT positive semidefinite.

Theorem.

(1) An real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive.
(2) An real symmetric matrix A is positive semidefinite if and only if all eigenvalues of A are non-negative.

Proof: A is symmetric if and only if $A = PDP^{-1} = PDP^T$ where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

So,

$$q(x_1, ..., x_n) = \vec{x}^T PDP^T\vec{x} = (P^T\vec{x})^T DP^T\vec{x}$$

Let $\vec{y} = P^T\vec{x}$, then

$$\begin{aligned} q(x_1, ..., x_n) &= \vec{x}^T PDP^T\vec{x} \\ &= (P^T\vec{x})^T DP^T\vec{x} \\ &= \vec{y}^T D\vec{y} \\ &= \lambda_1 y_1^2 + \cdots \lambda_n y_n^2. \end{aligned}$$

Proposition.

Let A be an $m \times n$ real matrix. Then A^TA is positive semidefinite. Further more, if $\text{rank}(A) = n$, then A^TA is positive definite.

Proof: $p(x_1, ..., x_n) = \vec{x}^T A^T A\vec{x} = (TA\vec{x})^T A\vec{x} = ||A\vec{x}||^2 \geq 0$. So, A^TA is positive semidefinite.
The equality hold if and only if $A\vec{x} = \vec{0}$. If $\text{rank}(A) = n$, $A\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$.
Hence, if $\text{rank}(A) = n$, then A^TA is positive definite.