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Definition. [Symmetric Matrices]

An $n \times n$ matrix A is called **symmetric** if $A^T = A$.
If we write $A = [a_{ij}]$, then A is symmetric if and only if

$$a_{ij} = a_{ji} \qquad \text{for all } i, j \in \{1, 2, ..., n\}$$

Example 1 (Diagonalizing a Symmetric Matrix). $A = \begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix}$.

1. $\det(A - \lambda I) = 0 \implies \lambda_1 = -2 \quad \lambda_2 = 13$

2. For $\lambda_1 = -2$, a basis for eigenspace $E_{\lambda_1} = \ker(A - \lambda_1 I)$ is $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$

   For $\lambda_2 = 13$, a basis for eigenspace $E_{\lambda_2} = \ker(A - \lambda_2 I)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

3. $A = HDH^{-1}$ $\quad D = \begin{bmatrix} -2 & 0 \\ 0 & 13 \end{bmatrix}$ $\quad H = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad H^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$

Example 2 (Diagonalizing a Symmetric Matrix). $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

1. $\lambda_1 = 0 \quad \lambda_2 = 3$

2. A basis for eigenbasis E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \xrightarrow{\text{Orthogonalize}} \cdots \longrightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

   A basis for eigenbasis E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

3. $A = PDP^{-1}$ where $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Example 3 (Diagonalizing a Symmetric Matrix). $A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}$.

1. $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 7 \\ 1 & 7-\lambda & 1 \\ 7 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 7 & 1-\lambda \end{vmatrix} + 7\begin{vmatrix} 1 & 7-\lambda \\ 7 & 1 \end{vmatrix}$

$= (1-\lambda)(\lambda^2-8\lambda+6) - (-\lambda-6) + 7(7\lambda-48)$

$= -\lambda^3+9\lambda^2+36\lambda-324$

$\lambda = 9,\ 6,\ -6$

2. An eigenvector for $\lambda=9$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

An eigenvector for $\lambda=6$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

An eigenvector for $\lambda=-6$ is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3. $A = HDH^{-1}$ $\quad H = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

Theorem.

Let A be a symmetric matrix and let λ, μ be two distinct eigenvalues of A with associated eigenvectors $\vec{v}, \vec{w}$. Then
$$\vec{v} \cdot \vec{w} = 0.$$

Definition. [Orthogonal Diagonalization]

An $n \times n$ matrix is **orthogonally diagonalizable** if there exist $n \times n$ matrices D and U, with D diagonal and U orthogonal (ie. $U^T U = I_n$), and with

$$A = UDU^{-1} = UDU^T.$$

Theorem. [On Orthogonal Diagonalizability]

An $n \times n$ matrix A is orthogonally diagnonalizable if and only if A is a symmetric matrix.

Theorem. [Spectral Theorem for Symmetric Matrices]

Evey $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real, and there are exactly n of them if counted with their multiplicities.
2. The dimension of the eigenspace E_λ associated to the eigenvalue λ, equals precisely the algebraic multiplicity of λ.
3. E_λ is orthogonal to E_μ for distinct eigenvalues λ, μ (in that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v} \in E_\lambda$ and $\vec{w} \in E_\mu$).
4. A is orthogonally diagonalizable.

Example 4 (Orthogonal Diagonalization). $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$A = PDP^{-1}$ where $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$A = UDU^{-1}$ $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & -1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & 0 & 2/\sqrt{6} \end{bmatrix}$

Example 5. (True or False)

1. If A and B are diagonalizable, then $A + B$ is diagonalizable.

False. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ are diagonalizable. But $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

2. If A and B are orthogonally diagonalizable, then $A+B$ is orthogonally diagonalizable.

True. Reason?

Example 6 (Orthogonal Diagonalization). $A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}$.

$A = HDH^{-1}$ $\quad H = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

$A = UDU^{-1}$ $\quad U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

3. The Spectral Decomposition

Let A be an $n \times n$ matrix and let D and U be a diagonal and orthogonal matrix with $A = UDU^{-1}$. Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of D, and let $\vec{u}_1, \dots, \vec{u}_n$ be the column vectors of U. Note that $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$.

A new decomposition of A in terms of $\lambda_1, \dots, \lambda_n$ and $\vec{u}_1, \dots, \vec{u}_n$ can be found by starting with the relation $A = UDU^{-1}$.

$$\begin{aligned} A &= UDU^{-1} = UDU^T \\ &= [\vec{u}_1 \; \dots \; \vec{u}_2] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} (\vec{u}_1)^T \\ \vdots \\ (\vec{u}_n)^T \end{bmatrix} \\ &= [\lambda_1 \vec{u}_1 \; \dots \; \lambda_n \vec{u}_n] \begin{bmatrix} (\vec{u}_1)^T \\ \vdots \\ (\vec{u}_n)^T \end{bmatrix} \\ &= \lambda_1 \vec{u}_1 \cdot (\vec{u}_1)^T + \dots + \lambda_n \vec{u}_n \cdot (\vec{u}_n)^T \end{aligned}$$

Theorem. [Spectral Decomposition for Symmetric Matrices]

Using above notations

$$A = \lambda_1 \Big(\vec{u}_1 \cdot (\vec{u}_1)^T \Big) + \cdots + \lambda_n \Big(\vec{u}_n \cdot (\vec{u}_n)^T \Big)$$

For each $i = 1, ..., n$, the matrix $\vec{u}_i \cdot (\vec{u}_i)^T$ is the projection matrix onto the line $\text{Span}(\vec{u}_i)$ in the sense that

$$\text{proj}_{\vec{u}_i}(\vec{x}) = \Big(\vec{u}_i \cdot (\vec{u}_i)^T \Big) \cdot \vec{x}, \qquad \text{for each } \vec{x} \in \mathbb{R}^n.$$

Example 7 (Spectral Decomposition for Symmetric Matrices)**.**

Ex 1 $A = \begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix}$ $A = UDU^{-1}$ $U = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$A = -2\vec{u}_1 \vec{u}_1^T + 13\vec{u}_2 \vec{u}_2^T$ $D = \begin{bmatrix} -2 & \\ & 13 \end{bmatrix}$

Ex2 $A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 7 & 1 \\ 7 & 1 & 1 \end{bmatrix}$

$= 9\vec{u}_1 \vec{u}_1^T + 6\vec{u}_2 \vec{u}_2^T + (-6)\vec{u}_3 \vec{u}_3^T$