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Theorem.

Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue for A if and only if the matrix equation

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

has a *nontrivial* solution $\vec{x}$.
Said differently, λ is an eigenvalue for A if and only if

$$\text{Nul}(A - \lambda I_n) \neq \{\vec{0}\}.$$

Definition.

Let A be an $n \times n$ matrix and λ be a eigenvalue of A. The set of all eigenvectors of A corresponding to λ together with the zero vector, is called the **eigenspace** of A corresponding to λ, and it equals the subspace

$$\text{Nul}(A - \lambda I_n).$$

The dimension of $\text{Nul}(A - \lambda I_n)$ is called the **geometric multiplicity** of λ.

$$1 \leq \text{Geometric multiplicity of } \lambda \leq \text{Algebraic multiplicity of } \lambda \leq n\ .$$

Example 1. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$. Explain the geometric meaning of the eigenvalues and eigenspaces.

From §7.1,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

For $\lambda = 1$ the eigenspace $E_{\lambda=1} = L$.
For $\lambda = 0$ the eigenspace $E_{\lambda=0} = L^{\perp}$.

Example 2. Find all eigenvalues and the corresponding eigenspaces of $A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 2 & 1-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{vmatrix} = \cdots = -(\lambda-2)^2(\lambda-3)$$

$\lambda=2$ $A-2I=\begin{bmatrix}2 & -1 & 0\\ 2 & -1 & 0\\ 2 & -1 & 0\end{bmatrix}\to\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ $x_1-\frac{1}{2}x_2=0$

x_2, x_3 free

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}x_2\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

A basis for the eigenspace of A corresponding to $\lambda=2$ is $\left\{\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$\lambda=3$ $A-3I=\begin{bmatrix}1 & -1 & 0\\ 2 & -2 & 0\\ 2 & -1 & -1\end{bmatrix}\to\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 0\\ 0 & 1 & -1\end{bmatrix}\to\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ $x_1=x_3$

$x_2=x_3$

x_3 free

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

A basis for the eigenspace of A corresponding to $\lambda=3$ is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

Theorem.

Let A be an $n\times n$ matrix and let $\vec{v}_1,\dots,\vec{v}_p$ be eigenvectors of A that correspond to *distinct* eigenvalues $\lambda_1,\dots,\lambda_p$ respectively. Then $\{\vec{v}_1,\dots,\vec{v}_p\}$ is a linearly independent set of vectors.

Theorem. [Review: Diagonalizability]

Let A be an $n\times n$ matrix.
A is diagonalizable if and only if it has n linearly independent eigenvectors. (eigenbasis.)
In this case $A=PDP^{-1}$ where the columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

Example 3. Diagonalizing Matrix $A=\begin{bmatrix}4 & -1 & 0\\ 2 & 1 & 0\\ 2 & -1 & 2\end{bmatrix}$

$$P=\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Proposition. [Case of Distinct Eigenvalues]

If an $n \times n$ matrix A has n **distinct** eigenvalues, then its corresponding eigenvectors are linearly independent and accordingly A is diagonalizable.

Theorem. [Case of Repeated Eigenvalues]

Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \ldots, \lambda_p$ such that

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{k_1}(\lambda_2 - \lambda)^{k_2} \cdots (\lambda_p - \lambda)^{k_p}.$$

Suppose $k_1 + k_2 + \cdots + k_p = n$. Let E_k be the eigenspace of λ_k. Suppose B_k is a basis for E_k.

- A is diagonalizable if and only if $B = B_1 \cup \cdots \cup B_p$ is an eigen-basis for A.
- A is diagonalizable if and only if

$$\dim E_1 + \cdots + \dim E_p = n.$$

This equality is satisfied if and only if $\dim(E_i) = k_i$ for each $i = 1, \ldots, p$

Example 4. Diagonalizing Matrix $A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

$$D=\begin{bmatrix} 3 & & \\ & 3 & \\ & & 6 \end{bmatrix} \quad P=\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Theorem. [Non Diagonalizing Result]

For any $n > 1$ there exist examples of $n \times n$ matrices that are not diagonalizable.

Example 5. For any $n > 1$, find examples of $n \times n$ non-diagonalizable matrices.

$$\begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix} \quad \begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix} \quad \begin{bmatrix} k & 1 & & \\ & k & 1 & \\ & & k & 1 \\ & & & k \end{bmatrix} \quad \cdots\cdots$$

Example 6. Diagonalizing the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. We already know that 1, -1, 4 are eigenvalues of A.

For $\lambda=1$

$$A-\lambda I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda=-1$

$$A-\lambda I = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda=4$

$$A-\lambda I = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 4 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example 7. Diagonalizing the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. We already know that 3 and 5 are eigenvalues of A.

$$D = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$