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Let A be an $n \times n$ matrix.

- Recall that a (possibly comlex) scalar λ is an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ has a nonzero solution.
- Equivalently, $(A - \lambda I_n)\vec{x} = \vec{0}$ has a nonzero solution.
- Equivalently, $A - \lambda I_n$ is not invertible.
- Equivalently, the determinant of $A - \lambda I_n$ equals zero.

Theorem. [The Characteristic Equation]

Let A be an $n \times n$ matrix. A (possibly comlex) scalar λ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

This last equation is called the **characteristic equation** of A.

Example 1. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ 3 & 4-\lambda \end{vmatrix}$

$= (2-\lambda)(4-\lambda) - 15$

$= \lambda^2 - 6\lambda - 7$

$= (\lambda-7)(\lambda+1) = 0$

$\lambda = 7 \quad \lambda = -1$

$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix}$

$= (2-\lambda)(4-\lambda) + 1$

$= \lambda^2 - 6\lambda + 9$

$= (\lambda-3)^2 = 0$

$\lambda = 3$

Theorem.

The eigenvalues of a (upper or lower) triangular $n \times n$ matrix A equal the diagonal entries of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \lambda = a_{11} \quad \lambda = a_{22} \quad \lambda = a_{33}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & * & * \\ 0 & a_{22}-\lambda & * \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

Example 2. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 & \sqrt{2} \\ 3 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 & \sqrt{2} \\ 3 & 4-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 2-\lambda & 5 \\ 3 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(2-\lambda)(4-\lambda) - 15\right]$$

$$= (3-\lambda)(\lambda^2 - 6\lambda - 7)$$

$$= (3-\lambda)(\lambda-7)(\lambda+1) = 0$$

$\lambda = 3 \quad \lambda = 7 \quad \lambda = -1$

2

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & 4-\lambda & 0 \\ 3 & 5 & 7-\lambda \end{vmatrix}$$

$$= (7-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix}$$

$$= (7-\lambda)(\lambda^2 - 6\lambda + 9)$$

$$= (7-\lambda)(\lambda-3)^2 = 0$$

$\lambda = 7 \quad \lambda = 3$

twice

Definition. [Characteristic Polynomial]

If A is an $n \times n$ matrix, the degree n polynomial

$$f_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of A.

Example 3. A and A^T have the same characteristic polynomial.

$$f_A(\lambda) = \det(A - \lambda I_n) = \det[(A - \lambda I_n)^T] = \det[(A^T - \lambda I_n)] = f_{A^T}(\lambda).$$

Example 4. Find the characteristic polynomial for a 2×2 arbitrary matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \underline{(a+d)}\lambda + \underline{ad - bc}$$

Definition.

Th sum of the diagonal entries of a square matrix is called the **trace** of A, denoted by $\operatorname{tr} A$.

Some basic properties about trace $\operatorname{tr} A$:

Proposition.

For $n \times n$ matrices A and B,

1. $\operatorname{tr}(A) = \operatorname{tr}(A^T)$
2. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
3. $\operatorname{tr}(kA) = k \operatorname{tr}(A)$
4. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Summarize Example 2: the characteristic polynomial for a 2×2 A:

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det(A)$$

More generally,

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & * & * & \cdots & * \\ * & a_{22}-\lambda & * & \cdots & * \\ * & * & a_{33}-\lambda & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & a_{nn}-\lambda \end{vmatrix}$$

Theorem.

Let A be an $n \times n$ matrix. Then the characteristic polynomial of A is

$$f_A(\lambda) = \det(A - \lambda I) = (-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \cdots + \det(A).$$

Definition. [Algebraic Multiplicity]

An eigenvalue λ_0 of A is said to have **algebraic multiplicity** k if it has multiplicity k as a root of the characteristic polynomial $f_A(t)$. Equivalently,

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

such that $g(\lambda_0) \neq 0$.

Example 5. Find all eigenvalues and their algebraic multiplicities of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ \lambda & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ \lambda & -\lambda \end{vmatrix}$$

$\lambda=0$ am=2

$\lambda=3$ am=3

$$= (1-\lambda)\left(-\lambda(1-\lambda)-\lambda\right) - (-2\lambda)$$

$$= (1-\lambda)(\lambda^2-2\lambda) + 2\lambda = \lambda^2-2\lambda-\lambda^3+2\lambda^2+2\lambda = 3\lambda^2-\lambda^3 = \lambda^2(3-\lambda) = 0$$

Theorem.

An $n \times n$ matrix has **at most** n real eigenvalues, even counted with algebraic multiplicities.

Example 6. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Which of the following numbers 1, −1, 4 are eigenvalues of A?

We only need to check $\det(A - I)$, $\det(A + I)$, $\det(A - 4I)$. All equal to zero. So, all 1, −1, 4 are eigenvalues.

A direct calculation of the eigenvalues is

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 2-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right] - (1-\lambda-2) + 2(1-2(2-\lambda))$$

$$= (1-\lambda)(\lambda^2-3\lambda+1) + 5(\lambda-1)$$

$$= (1-\lambda)(\lambda^2-3\lambda-4)$$

$$= (1-\lambda)(\lambda+1)(\lambda-4)$$

Example 7. Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. Verify that 3 and 5 are eigenvalues.

$$f(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ -1 & 4-\lambda & 2 \\ -1 & 1 & 5-\lambda \end{vmatrix}$$

$$f(3) = \begin{vmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 0$$

$$f(5) = \begin{vmatrix} -3 & 1 & 2 \\ -1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$$

Theorem.

Let A be an $n \times n$ matrix. Suppose A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (listed with algebraic multiplicities.) Then

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

and

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

This theorem comes from

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Theorem. [On Eigenvalues of Similar Matrices]

If A and B are similar, i.e., $A = PBP^{-1}$, then they have the same characteristic polynomial, i.e. $f_A(\lambda) = f_B(\lambda)$, and hence the same eigenvalues with the same multiplicities.

$f_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I) = f_B(\lambda)$. So, A and B have the same characteristic polynomial.

The converse of the preceding theorem is not generally true. There do exist matrices with the same characteristic polynomial that are not similar matrices. See example later.

Proposition. Determinant and trace of similar matrices

If A and B are similar, we also have $\det(A) = \det(B)$, $\operatorname{tr}(A) = \operatorname{tr}(B)$.

Proof. Since determinant and trace are determined by characteristic polynomial, so we get the result by the above theorem.

Proposition. rank of similar matrices

If A and B are similar, then $\operatorname{rank}(A) = \operatorname{rank}(B)$.

$A = PBP^{-1}$. Multiplying an invertible matrix does not change the rank. So, $\text{rank}(A) = \text{rank}(B)$.

Example 8. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar, but they have the same rank, the same determinant, the same trace, the same eigenvalues.

If B similar to $A = I$, then $B = PAP^{-1} = I$ which is a contradiction.

Example 9. Are the following two matrices similar to each other? $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$

$\text{tr}(A) = 5$ but $\text{tr}(B) = 6$
$|A| = 2$ but $|B| = -1$

Warning: Similar matrices may have **different** eigenvectors.

Think about Example 1 in §7.1. The projection matrix $A = \dfrac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of D but it is not an eigenvector of A.