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1. Diagonalization

Let D be an diagonal matrix. The power D^k is easy to calculate. For example,

$$D^k = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}^k = \begin{bmatrix} (d_1)^k & 0 & 0 & 0 \\ 0 & (d_2)^k & 0 & 0 \\ 0 & 0 & (d_3)^k & 0 \\ 0 & 0 & 0 & (d_4)^k \end{bmatrix}$$

Definition.

An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix D, that is, if there exists an invertible matrix P such that $A = PDP^{-1}$.

Powers of a diagonalizable matrix A are also easy to calculate:
$A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^kP^{-1}$

$$A^k = PD^kP^{-1}$$

We see that A^k is similar to the diagonal matrix D^k, and hence also **diagonalizable**.

Question:

1. Are all $n \times n$ matrices A diagonalizable?

2. If a matrix A is diagonalizable, how to find the invertible matrix P and the diagonal matrix D? The answer for this question is called **diagonalize** matrix A.

Solve $A = PDP^{-1}$. That is $AP = PD$. More explicitly (when $n = 3$)

$$A[\vec{b}_1\ \vec{b}_2\ \vec{b}_3] = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3] \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

That is $[A\vec{b}_1\ A\vec{b}_2\ A\vec{b}_3] = [d_1\vec{b}_1\ d_2\vec{b}_2\ d_3\vec{b}_3]$
So, equivalently, we need to find numbers d_1, d_2, d_3 and $\vec{b}_1, \vec{b}_2, \vec{b}_3$ satisfy
$A\vec{b}_1 = d_1\vec{b}_1$, $A\vec{b}_2 = d_2\vec{b}_2$, $A\vec{b}_3 = d_3\vec{b}_3$. They are the same equation:

$$A\vec{x} = d\vec{x}$$

Recall from §3.4 the meaning of similar matrices $A = PDP^{-1}$. (If you have not learned §3.4, you can skip this page.)

Let A be the matrix of a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$ and denote $P = [\vec{b}_1 \ldots \vec{b}_n]$ the change of coordinate matrix. The *matrix of T respect to basis $\mathscr{B}$* is

$$D = \Big[[T(\vec{b}_1)]_{\mathscr{B}}\ [T(\vec{b}_2)]_{\mathscr{B}}\ \cdots\ [T(\vec{b}_n)]_{\mathscr{B}}\Big]$$

Then, $A = PDP^{-1}$.

Example 1. (Example 6 in §3.4) Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix}1\\2\\3\end{bmatrix}\}\ \mathbb{R}^3$. The matrix of T is $A = \dfrac{1}{14}\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$

(1)Find a basis $\mathscr{B} = \{\vec{b}_1\ \vec{b}_2\ \vec{b}_3\}$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is the diagonal matrix $D = \begin{bmatrix}d_1&0&0\\0&d_2&0\\0&0&d_3\end{bmatrix}$. (2) Equivalently, find a matrix $B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ and D such that $A = PDP^{-1}$.

Step 1. Compare the columns of D. It is equivalent to find independent vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and numbers d_1, d_2, d_3 such that

$$T(\vec{b}_1) = d_1(\vec{b}_1), \quad T(\vec{b}_2) = d_2(\vec{b}_2), \quad T(\vec{b}_2) = d_2(\vec{b}_2)$$

Step 2. Use the geometric properties of the transformation to find those vectors and numbers. (We will develop algebraic method to solve this systematically.)

We need to find vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ such that the projection $\text{proj}_L\, \vec{b}_i$ is the scalar product of $\vec{b}_i$.

Let $\vec{b}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Then, $A\vec{b}_1 = 1\vec{b}_1$. So, $d_1 = 1$.

Let $\vec{b}_1 = \begin{bmatrix}2\\-1\\0\end{bmatrix}$. Then, $A\vec{b}_2 = \vec{0} = 0\vec{b}_2$. So, $d_2 = 0$.

Let $\vec{b}_1 = \begin{bmatrix}3\\0\\-1\end{bmatrix}$. Then, $A\vec{b}_3 = \vec{0} = 0\vec{b}_3$. So, $d_3 = 0$.

The key is to solve $T(\vec{x}) = \lambda\vec{x}$ or equivalently $A\vec{x} = \lambda\vec{x}$.

2. Eigenvalues and Eigenvectors.

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by matrix $T\vec{x} = A\vec{x}$.

Definition.

- An **eigenvector** of A is a nonzero n-dimensional vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}$$

  for some (possibly complex) scalar λ.

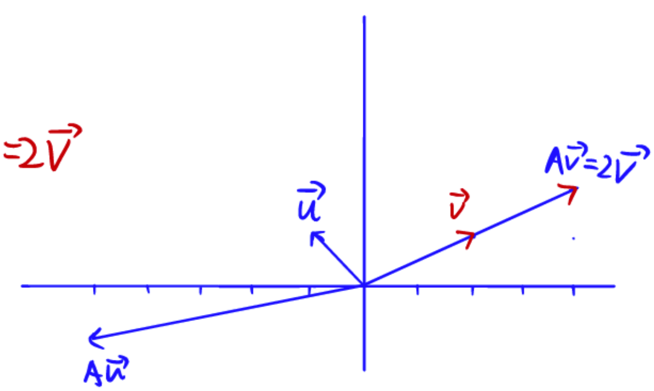
- An **eigenvalue** of A is a (possibly complex) scalar λ for which there exists a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. We say that $\vec{x}$ is **an eigenvector corresponding to** λ.

- A basis $\vec{b}_1, \ldots, \vec{b}_n$ of $\mathbb{R}^n$ is called an **eigenbasis** for A if the vectors $\vec{b}_1, \ldots, \vec{b}_n$ are eigenvectors of A.

Example 2. Geometric meaning of eigenvalue and eigenvector. $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A\vec{u} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$A\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\vec{v}$



Example 3. If $\vec{v}$ is an eigenvector of A corresponding to λ, is $\vec{v}$ an eigenvector of A^k?

$A\vec{v} = \lambda\vec{v}$. Hence,

$A^k\vec{v} = A^{k-1}A\vec{v} = A^{k-1}\lambda\vec{v} = \lambda A^{k-1}\vec{v} = \cdots = \lambda^k\vec{v}$

$\vec{v}$ is an eigenvector of A^k corresponding to eigenvalue λ^k

Example 4. If $\vec{v}$ is an eigenvector of an invertible matrix A corresponding to λ, is $\vec{v}$ an eigenvector of A^{-1}?

$A\vec{v} = \lambda\vec{v}$. Hence, $A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$. So, $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$. So, $\vec{v}$ an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

Example 5. Consider the eigenvalue of a polynomial of A. For example, what is the eigenvalue of $3A^3 - 2A^2$?

Theorem.

The matrix A is diagonalizable if and only if A has an eigenbasis $\vec{b}_1, \ldots, \vec{b}_n$, i.e., n independent eigenvectors.
In this case $A = PDP^{-1}$ where $P = [\vec{b}_1 \ \ldots \ \vec{b}_n]$; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

Proof: We already verified that system of equations $A\vec{b}_1 = \lambda_1\vec{b}_1$, $A\vec{b}_2 = \lambda_2\vec{b}_2$, $\ldots$, $A\vec{b}_n = \lambda_n\vec{b}_n$. is equivalent to matrix equation

$$AP = PD$$

where $P = [\vec{b}_1 \ \ldots \ \vec{b}_n]$ and $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$.

P is invertible if and only if $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of $\mathbb{R}^n$. In this case, $A = PDP^{-1}$ and A is diagonalizable.

Example 6. Write down all matrices A, P and D in Example 1.

$\vec{a}_1 = \text{proj}_{\vec{v}}\vec{e}_1 = \left(\frac{\vec{e}_1\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{1}{14}\right)\begin{bmatrix}1\\2\\3\end{bmatrix}$

$\vec{a}_2 = \text{proj}_{\vec{v}}\vec{e}_2 = \frac{2}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$

$\vec{a}_3 = \frac{3}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$

$T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$T\left(\begin{bmatrix}2\\-1\\0\end{bmatrix}\right) = 0\begin{bmatrix}2\\-1\\0\end{bmatrix}$

$T\left(\begin{bmatrix}3\\0\\-1\end{bmatrix}\right) = 0\begin{bmatrix}3\\0\\-1\end{bmatrix}$

$A = \begin{bmatrix} 1/14 & 2/14 & 3/14 \\ 2/14 & 4/14 & 6/14 \\ 3/14 & 6/14 & 9/14 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A = PDP^{-1}$.

Example 7. Let T be the rotation through an angle of $\pi/2$ in the counterclock direction. So the matrix of T is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A. Is A diagonalizable?

Rotation a vector of $\pi/2$ counterclockwise can never be its scalar product.

So, there is no eigenvectors.

So A is not diagonalizable.

Example 8. Find all possible real eigenvalues of an $n \times n$ orthogonal matrix.

$\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$

If $\vec{v}$ is an eigenvector of A corresponding eigenvalue λ. then $A\vec{v} = \lambda\vec{v}$.

$\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda| \|\vec{v}\| \Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1$

Example 9. Which matrix has 0 as an eigenvalue?

$\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$

If $\vec{v}$ is an eigenvector of A corresponding eigenvalue λ. then $A\vec{v} = \lambda\vec{v}$.

$\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda| \|\vec{v}\| \Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1$

Proposition.

A has 0 as a eigenvalue if and only if A is not invertible.