• Instructor: He Wang Email: he.wang@northeastern.edu

1.How Row Operations Affect the Determinant

Recall that there are three types of *elementary row operations*:
1. (Replacement) Add to one row the multiple of another row. $(R_i + kR_j)$
2. (Interchange) Interchange two rows. $R_i \leftrightarrow R_j$
3. (Scaling) Multiply all entries of a given row by a nonzero constant. cR_i for $c \neq 0$.

Question:
If an $n \times n$ matrix A is modified by a single one of the elementary row operations, how does that affect its determinant?

Theorem. [Row Operations and the Determinant]

Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.
1. If B is obtained from A by a Replacement operation, then

$$\det B = \det A.$$

2. If B is obtained from A by an Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Scaling operation by a factor k, then

$$\det B = k \det A.$$

Theorem.

Let A be an $n \times n$ matrix.

$$\det(kA) = (k^n)(\det A).$$

For example $\det(kA) = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3(\det(A))$

Theorem.

Let A be an $n \times n$ matrix that can be reduced to a matrix U in echelon form with only Replacement and Interchange operations. Then

$$\det A = (-1)^r \cdot \det U$$

where r is the number of Interchange operations used to get from A to U.

• The determinant $\det U = 0$ if and only if U has a 0 on its diagonal, which in turn can only happen if U has a row of zeros.

Theorem. [Determinant of the Transpose Matrix]

$$\det A^T = \det A.$$

Theorem. [Determinants of Products of Matrices]

$$\det(AB) = (\det A)(\det B).$$

$$\det(A^m) = (\det(A))^m$$

Theorem.

Let A be an $n \times n$ invertible matrix.

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Question: How about $\det(A + B)$? Is it $\det(A) + \det(B)$?

Think about $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Example 1. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Is A invertible?

$$\det A \xlongequal[R_3-R_1]{R_2-R_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 13 & 14 & 15 & 16 \end{vmatrix} \xlongequal{R_3-2R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0$$

So, A is not invertible.

Example 2. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 11 & 14 \\ 7 & 8 & 15 & \pi \end{bmatrix}$. Is A invertible?

$$\det A \xlongequal[R_3-3R_1]{R_2-2R_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 7 & 8 & \sqrt{3} & \pi \end{vmatrix} = (-1)^{2+4} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 7 & 8 & \sqrt{3} \end{vmatrix} = 2(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= -2(8-14) = 12$$

So, A is invertible.

2. Column Operations

Definition. [Elementary Column Operations]

1. (Column Replacement) Add to one column the multiple of another column.
2. (Column Interchange) Interchange two columns.
3. (Column Scaling) Multiply all entries of a given column by

Since column operations on a matrix A can be thought of as row operations on its transpose matrix A^T , and since

$$\det A = \det A^T,$$

the rules for how elementary row operations affect the determinant can be used to give a similar rule for column operations.

Theorem. [Column Operations and the Determinant]

Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.

1. If B is obtained from A by a Column Replacement operation, then

$$\det B = \det A.$$

2. If B is obtained from A by a Column Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Column Scaling operation by a factor k, then

$$\det B = k \det A.$$

3. A Linearity Property of the determinant function

Let A be an $n \times n$ matrix with column vectors $\vec{a}_1, \cdots, \vec{a}_n$,

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

Let $\vec{x}$ be an n-dimensional vector (an $n \times 1$ matrix) and consider the transformation

$$T_j \colon \mathbb{R}^n \to \mathbb{R}$$

defined by

$$T_j(\vec{x}) = \det(\begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_{j-1} & \vec{x} & \vec{a}_{j+1} & \dots & \vec{a}_n \end{bmatrix})$$

Theorem. [Linearity and Determinants]

The transformation T_j defined above is a linear transformation, that is
(a) $T_j(\vec{x} + \vec{y}) = T_j(\vec{x}) + T_j(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and
(b) $T_j(c\vec{x}) = cT_j(\vec{x})$ for all $\vec{x} \in R^n$ and all $c \in R$.
Accordingly there exists a $1 \times n$ matrix B such that $T_j = T_B$.

Example 3. Finding the matrix B for the linear transformation T_2 for a given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$T_2(\vec{x}) = \begin{bmatrix} a_{11} & x_1 & a_{13} \\ a_{21} & x_2 & a_{23} \\ a_{31} & x_3 & a_{33} \end{bmatrix}$$
$$= x_1 C_{12} + x_2 C_{22} + x_3 C_{32}$$
$$= [C_{12}\ C_{22}\ C_{32}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= B\vec{x}$$

So, the matrix B for the linear transformation T_2 is $[C_{12}\ C_{22}\ C_{32}]$
Here, C_{ij} is the (i, j)-th cofactor defined as $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Cramer's Rule

Consider a matrix equation $A\vec{x} = \vec{b}$ in which A is an $n \times n$ invertible matrix, and write $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$. Let

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \dots & \vec{a}_n \end{bmatrix}$$

Theorem. [Cramer's Rule]

The unique solution $\vec{x}$ of the matrix equation $A\vec{x} = \vec{b}$ (for the case when A is an $n \times n$ invertible matrix), is given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}, \quad \text{for } i = 1, 2, \dots, n.$$

First, from cofactor expansion, $\det(A_i(\vec{b})) = \sum_{j=1}^{n} b_j C_{ij}$.

$$\begin{aligned} a_{k1}x_1 + a_{k2}x_2 + \cdots a_{kn}x_n &= \frac{1}{\det(A)}(\sum_{i=1}^{n} a_{ki} \sum_{j=1}^{n} b_j C_{ij}) \\ &= \frac{1}{\det(A)}(\sum_{j=1}^{n} b_j \sum_{i=1}^{n} a_{ki} C_{ij}) \\ &= \frac{1}{\det(A)}(b_k \det(A)) \\ &= b_k \end{aligned}$$

for any $k = 0, 1, \cdots, n$. This verifies that $(x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b}$.

Let C be the associated $n \times n$ matrix of cofactors defined as:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

The transpose of C is called the **adjugate matrix of** A, denoted by $\mathrm{adj}A$:

$$\mathrm{adj}A = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Theorem.

If A is a invertible matrix then $A^{-1} = \dfrac{1}{\det A} \cdot \mathrm{adj}A$

Sometimes, the following formula is also used in the definition of determinant. It is not easy to use it for computation, but it is useful for prove some theorems we learned in the chapter.

If A is an $n \times n$ matrix, then

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S(n)} \pm \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma \in S(n)} \pm a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}. \end{aligned}$$

Here σ is a permutation and the sign is determined by the number of inversions of σ.

Example 4. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The $3! = 6$ permutations of [3] are listed below, along with the determinant of the associated permutation matrix; For the 6 permutations $(\sigma(1)\ \sigma(2)\ \sigma(3))$,

$$\begin{array}{ll} (1\ \ 2\ \ 3) & sign = \ \ 1 \\ (1\ \ 3\ \ 2) & sign = -1 \\ (2\ \ 1\ \ 3) & sign = -1 \\ (2\ \ 3\ \ 1) & sign = \ \ 1 \\ (3\ \ 1\ \ 2) & sign = \ \ 1 \\ (3\ \ 2\ \ 1) & sign = -1 \end{array}$$

Hence we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Geometric meaning of determinant

2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The determinant $\det(A) =$ the **area** of the parallelogram spanned by $\vec{a}_1, \vec{a}_2$.

3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The determinant $\det(A) =$ the **volume** of the parallelepiped spanned by $\vec{a}_1, \vec{a}_2, \vec{a}_3$.