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Recall that the *determinant* of a 2×2 matrix is given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

We expand this definition to 1×1 matrices by setting

$$\det \begin{bmatrix} a \end{bmatrix} = a.$$

For 1 and 2×2 matrices, we have the following property:

A is invertible if and only if $\det A \neq 0$.

Goal: Define the determinant of an $n \times n$ matrix A with $n \geq 3$, such that A is invertible if and only if $\det A \neq 0$.

Definition.

Let A be an $n \times n$ matrix with $n \geq 2$ and with (i, j)-th entry a_{ij} .
Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column from A.
Then the **determinant of** A, denoted $\det A$, is defined as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

This formula for $\det A$ is called the **first row cofactor expansion** formula for the determinant of A.

Example 1. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Theorem.

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Example 2. Find the determinant of $A = \begin{bmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$. Is A invertible?

$\det A = \begin{vmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{vmatrix} = 0\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} - 4\begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 5 & 2 \\ 0 & 2 \end{vmatrix}$

$= -4(-5) + 2(10)$

$= 40$ $\quad \det A \neq 0 \Rightarrow A$ is invertible.

Example 3. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 6 & 7 \end{bmatrix}$. Is A invertible?

$\det A = 1\begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 \\ 5 & 6 \end{vmatrix} = -24 - 2(-20) + 3(0) = 16$

$\det A \neq 0 \Rightarrow A$ is invertible.

Definition.

Let A be an $n \times n$ matrix. Its (i, j)**-th cofactor** C_{ij} is defined as

$$C_{ij} = (-1)^{i+j} \cdot \det A_{ij}$$

where, as before, A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i-th row and j-th column.

Using cofactors, the first row cofactor expansion formula for the determinant of A can be rewritten as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Theorem.

The determinant of an $n \times n$ matrix A can be computed via cofactor expansions across any row or down any column of A:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

These formulas are called the **i-th row** and **j-th column** cofactor expansions for $\det A$, respectively.

Example 4. Redo Examples 1 and 2.

Ex 1 $\det A = 5C_{21} = 5(-1)^{2+1}\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = 5(-1)(-8) = 40$

Ex 2 $\det A = 4C_{32} = 4(-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 4(-1)(-4) = 16$

Example 5. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 1.2 \\ 0 & 0 & 0 & 2 \\ 5 & 6 & 7 & \pi \\ 0 & 1 & 2 & \sqrt{2} \end{bmatrix}$

$\det A = 2(-1)^{4+2}\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 0 & 1 & 2 \end{vmatrix}$

$= 2\left(\begin{vmatrix} 6 & 7 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\right)$

$= 2\left((12-7) - 5\right)$

$= 2(0)$

$= 0$

So, A is not invertible.

Recall the definition of **lower triangular matrix.** Similarly, we can define **upper triangular matrix.** An $n \times n$ matrix A is called **triangular** if it is either lower or upper triangular.

Theorem. [Determinants of Triangular Matrices]

Let A be an $n \times n$ triangular matrix, then $\det A$ equals the product of the diagonal entries of A:

$$\det A = a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

Example 6. Find the determinant of $A = \begin{bmatrix} 2 & \sqrt{2} & 3 & 1.7 \\ 0 & 3 & 7 & 12 \\ 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Is A invertible?

$\det(A) = 2(3)(1)(5) = 30$. So A is invertible.

Example 7. Find out for which value of λ the matrix $A - \lambda I$ is not invertible, where $A = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & \sqrt{2} & 1.7 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & \sqrt{2} & 1.7 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(5-\lambda) = 0 \quad \Rightarrow \lambda = 2, 3, 5$$

Example 8. If A is an $n \times n$ matrix. Consider the relation between $\det(kA)$, $\det(A^{-1})$, $\det(A^T)$ and $\det(A)$.
We consider this in the next section.

Determinant of a Block Matrix.

Theorem.

If $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, then,

$$\det(M) = \det(A)\det(C).$$

No such formula for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, in general.

Example 9.

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 2 & 11 & \sqrt{3} \\ 2 & 3 & \pi & 12 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\det M = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 9 \\ 1 & 4 \end{vmatrix}$$
$$= (-1)(3)$$
$$= -3$$