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In §2.2, we defined dot product, norm(length), orthogonal(perpendicular) and angle.

**Definition.** [Distance Between Vectors]

The **distance** $\text{dist}(\vec{u}, \vec{v})$ between vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is defined as

$$\text{dist}(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

Note that $\text{dist}(\vec{u}, \vec{v}) \geq 0$ for any pair of vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$\text{dist}(\vec{u}, \vec{v}) = 0$ if and only if $\vec{u} = \vec{v}$.

$\text{dist}(\vec{u}, \vec{v}) = \text{dist}(\vec{v}, \vec{u})$.

**Theorem.** [Pythagorean Theorem]

Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if and only if they satisfy the **Pythagorean Relation**

$$||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2.$$

Proof: Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.
$||\vec{u} + \vec{v}||^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} = ||\vec{u}||^2 + ||\vec{v}||^2 + 2\vec{u} \cdot \vec{v}$

**Theorem.** [Cauchy-Schwarz inequality]

$$|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \cdot ||\vec{y}||$$

Proof: First, we have $\vec{y} = \text{proj}_{\vec{x}}(\vec{y}) + \vec{y}^{\perp}$.

$$||\vec{y}||^2 = \vec{y} \cdot \vec{y} = ||\,\text{proj}_{\vec{x}}(\vec{y})||^2 + ||\vec{y}^{\perp}||^2 = \frac{(\vec{y} \cdot \vec{x})^2}{\vec{x} \cdot \vec{x}} + ||\vec{y}^{\perp}||^2 \geq \frac{(\vec{y} \cdot \vec{x})^2}{\vec{x} \cdot \vec{x}}$$

Hence $||\vec{y}||^2 ||\vec{x}||^2 \geq (\vec{x} \cdot \vec{y})^2$. That is $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \cdot ||\vec{y}||$.
The equality holds if and only if $\vec{y}^{\perp} = \vec{0}$.

Recall that

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \cdot ||\vec{y}||}$$

$|\vec{x}\cdot\vec{y}| = ||\vec{x}||\cdot||\vec{y}||$ if and only $\vec{x}$ and $\vec{y}$ in the same or opposite direction.

**Example 1.** Find the norm, angle, and distance between vectors $\vec{v}_1 = \begin{bmatrix}3\\1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}$

$||\vec{v}_1|| = \sqrt{9+1+1} = \sqrt{11}$ $||\vec{v}_2|| = \sqrt{1+4+1} = \sqrt{6}$

$\theta = \arccos \frac{\vec{v}_1\cdot\vec{v}_2}{||\vec{v}_1||\,||\vec{v}_2||} = \arccos\frac{0}{\sqrt{11}\sqrt{6}} = \frac{\pi}{2}$

$\vec{v}_1 - \vec{v}_2 = \begin{bmatrix}4\\-1\\0\end{bmatrix}$

distance $(\vec{v}_1, \vec{v}_2) = ||\vec{v}_1 - \vec{v}_2|| = \sqrt{16+1} = \sqrt{17}$

**Definition.** [Orthogonal Set]

A set $\{\vec{u}_1, \ldots, \vec{u}_p\}$ of vectors in $\mathbb{R}^n$ is called **orthogonal** if $\vec{u}_i \cdot \vec{u}_j = 0$ for any choice of indices $i \neq j$.

**Theorem.**

- Orthogonal vectors are linear independent.
- Orthonormal vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.

**Definition.**

- An **orthogonal basis** for a subspace $W$ of $\mathbb{R}^n$ is any basis for $W$ which is also an orthogonal set.
- If each vector is a **unit** vector in an orthogonal basis, then it is an **orthonormal basis**.

**Example 2.** Is the set $\mathscr{B} = \left\{ \vec{v}_1 = \begin{bmatrix}3\\1\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-1\\-4\\7\end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$? If it is, normalize this basis.

$\vec{v_1}\cdot\vec{v_2}=0$
$\vec{v_1}\cdot\vec{v_3}=0$
$\vec{v_2}\cdot\vec{v_3}=0$

So $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^3$

$$\vec{u_1}=\frac{\vec{v_1}}{\|\vec{v_1}\|}=\frac{1}{\sqrt{11}}\begin{bmatrix}3\\1\\1\end{bmatrix},\quad \vec{u_2}=\frac{\vec{v_2}}{\|\vec{v_2}\|}=\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix},\quad \vec{u_3}=\frac{\vec{v_3}}{\|\vec{v_3}\|}=\frac{1}{\sqrt{66}}\begin{bmatrix}-1\\-4\\7\end{bmatrix}$$

**Theorem.** [Coordinates with respect to an orthogonal basis]

Let $\mathscr{B}=\{\vec{u}_1,\ldots,\vec{u}_p\}$ be an **orthogonal** basis for a subspace $W$ of $\mathbb{R}^n$, and let $\vec{y}$ be any vector in $W$. Then

$$\vec{y}=\left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1+\cdots+\left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

If $W=\mathbb{R}^n$, then the $\mathscr{B}$-coordinates of $\vec{y}$ are given by:

$$[\vec{y}]_{\mathscr{B}}=\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}\quad\text{with}\quad c_i=\frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}=\frac{\vec{y}\cdot\vec{u}_i}{||\vec{u}_i||^2}$$

Let $\mathscr{B}=\{\vec{u}_1,\ldots,\vec{u}_p\}$ be an **orthonormal** basis for a subspace $W$ of $\mathbb{R}^n$, and let $\vec{y}$ be any vector in $W$. Then

$$\vec{y}=(\vec{y}\cdot\vec{u}_1)\,\vec{u}_1+\cdots+(\vec{y}\cdot\vec{u}_p)\,\vec{u}_p$$

**Example 3.** Find the coordinate of $\vec{y}=\begin{bmatrix}6\\1\\-8\end{bmatrix}$ relative basis of $\mathbb{R}^3$ in Example 1.

$$c_1 = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{-66}{66} = -1$$

So $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$

**Example 4.** Is the set $\mathscr{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$? If it is, normalize this basis. Find the coordinate of $\vec{y} = \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix}$ relative to $\mathscr{B}$.

$\vec{v}_1\cdot\vec{v}_2 = 0$
$\vec{v}_1\cdot\vec{v}_3 = 0$
$\vec{v}_2\cdot\vec{v}_3 = 0$
$\Rightarrow$ orthogonal basis.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$c_1 = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{6}{6} = 1$$

$$c_2 = \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{4}{2} = 2$$

$$c_3 = \frac{\vec{y}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{9}{3} = 3$$

So $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

We defined the Orthogonal Projection Onto A Line in §2.2:

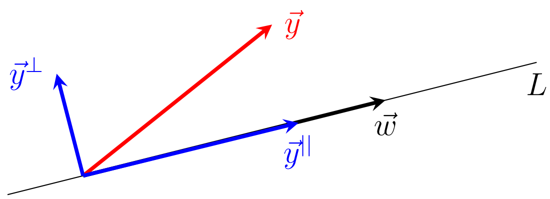
Let $L = \text{Span}\{\vec{w}\}$ be the line in $\mathbb{R}^n$ spanned by $\vec{w} \in \mathbb{R}^n$. For a given vector $\vec{y} \in \mathbb{R}^n$, the vector

$$y^{||} = \text{proj}_L(\vec{y}) := \left(\frac{\vec{y} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}$$

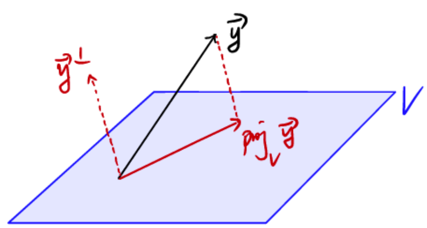
is called the **orthogonal projection of $\vec{y}$ onto** $L$ (or onto $\vec{w}$) and

$$\vec{y}^{\perp} := \vec{y} - \text{proj}_L(\vec{y})$$

**the component of $\vec{y}$ orthogonal to** $L$ (or $\vec{w}$).




We can define the orthogonal projection on to an plane.




More generally, given a subspace $W$ of $\mathbb{R}^n$ and a vector $\vec{y} \in \mathbb{R}^n$, we can ask if/how one can find a decomposition of $\vec{y}$ as

$$\vec{y} = \text{proj}_W(\vec{y}) + \vec{y}^{\perp}$$

with $\text{proj}_W(\vec{y}) \in W$ (**the orthogonal projection of $\vec{y}$ on to** $W$) and $\vec{y}^{\perp}$ is the **component of $\vec{y}$ perpendicular to** $W$.

**Theorem.** Orthogonal Decomposition

Let $W$ be any subspace of $\mathbb{R}^n$ and let $\vec{y} \in \mathbb{R}^n$ be any vector. Then there exists a unique decomposition

$$\vec{y} = \text{proj}_W(\vec{y}) + \vec{y}^{\perp}$$

with $\text{proj}_W(\vec{y}) \in W$ and $\vec{y}^{\perp}$ is perpendicular to $W$.

**Theorem.** Orthogonal Decomposition

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an *orthogonal basis* for $W$, then

$$\text{proj}_W(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \cdots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

and $\vec{y}^{\perp} = \vec{y} - \text{proj}_W(\vec{y})$.

**Example 5.** Let $V$ be a subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$. Find the orthogonal projection of $\vec{y} = \begin{bmatrix}1\\2\\6\end{bmatrix}$ on to $V$ and find the orthogonal component of $\vec{y}$.

$\vec{v_1}\cdot\vec{v_2}=0$ so orthogonal.

$$\text{proj}_V \vec{y} = \frac{\vec{y}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\cdot\vec{u_1} + \frac{\vec{y}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}}\vec{u_2} = \frac{5}{2}\begin{bmatrix}1\\1\\2\end{bmatrix} + \frac{-1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}2\\3\\5\end{bmatrix}$$

$$\vec{y}^{\perp} = \vec{y} - \text{proj}_V\vec{y} = \begin{bmatrix}1\\2\\6\end{bmatrix} - \begin{bmatrix}2\\3\\5\end{bmatrix} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

**Definition.** [Orthogonal Complements]

Given a nonempty **subset** (finite or infinite) $W$ of $\mathbb{R}^n$, we define its **orthogonal complement** $W^{\perp}$ (pronounced "$W$ perp") as the set of all vectors $\vec{v} \in \mathbb{R}^n$ such that

$$\vec{v}\cdot\vec{w} = 0, \qquad \text{for all } \vec{w} \in W.$$

Expressed in set notation:

$$W^{\perp} = \{\vec{v} \in \mathbb{R}^n \mid \vec{v}\cdot\vec{w} = 0 \text{ for all } \vec{w} \in W\}$$

**Example 6.** Find a basis for $V^{\perp}$, where $V = \text{Span}\left\{\begin{bmatrix}0\\0\\2\\8\end{bmatrix}, \begin{bmatrix}1\\5\\2\\-5\end{bmatrix}, \begin{bmatrix}2\\10\\6\\-2\end{bmatrix}\right\}$

$$V^\perp = \left\{ \vec{x} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \vec{x}\cdot\vec{v_1} = 0 \\ \vec{x}\cdot\vec{v_2} = 0 \\ \vec{x}\cdot\vec{v_3} = 0 \end{array} \right\}$$

$$= \left\{ \vec{x} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 2x_3 + 8x_4 = 0 \\ x_1 + 5x_2 + 2x_3 - 5x_4 = 0 \\ 2x_1 + 10x_2 + 6x_3 - 2x_4 = 0 \end{array} \right\}$$

$A = \begin{bmatrix} 0 & 0 & 2 & 8 \\ 1 & 5 & 2 & -5 \\ 2 & 10 & 6 & -2 \end{bmatrix}$ So, $V^\perp = \ker(A)$ which has a basis $\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

**Theorem.**

Let $W$ be a subset of $\mathbb{R}^n$. Let $V = \text{Span}(W)$, then

1. $V^\perp = W^\perp$
2. If $V = \text{Span}(\mathscr{B})$, then $V^\perp = \mathscr{B}^\perp$
3. $W^\perp$ is a subspace of $\mathbb{R}^n$ (even when $W$ is not).
4. $(V^\perp)^\perp = V$.
5. $\dim V + \dim V^\perp = n$.

Let $A$ be an $m \times n$ matrix.

The **row space** of $A$ is $\text{Row}(A)$, spanned by the row vectors of $A$.

The **column space** of $A$ is $\text{Col}(A)$, so $\text{Col}(A) = \text{im}(A)$.

The kernel of $A$ is also called the **null space** of $A$, denoted $\text{Nul}(A)$.

**Theorem.**

Let $A$ be an $m \times n$ matrix, then

$$(\text{Row } A)^\perp = \ker(A)$$

**Theorem.**

Let $A$ be an $m \times n$ matrix, then

$$(\operatorname{im} A)^{\perp} = \ker A^T.$$

**Example 7.** In Example 6, $A = \begin{bmatrix} 0 & 0 & 2 & 8 \\ 1 & 5 & 2 & -5 \\ 2 & 10 & 6 & -2 \end{bmatrix}$ then

$$V = \operatorname{Row} A = \operatorname{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 6 \\ -2 \end{bmatrix} \right\}$$

**Example 8.** Let $V$ be a subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

(1) Is $\{\vec{v}_1, \vec{v}_2\}$ an orthogonal basis for $V$?

Yes, since $\vec{v}_1 \cdot \vec{v}_2 = 0$.

(2). Find the decomposition of $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ as $\vec{y} = \vec{y}_1 + \vec{y}_2$ such that $\vec{y}_1 \in V$ and $\vec{y}_2 \in V^{\perp}$.

$$\vec{y}_1 = \operatorname{proj}_V \vec{y} = \left(\frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2 = \left(\frac{1}{9}\right)\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{11}{9}\right)\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 13/9 \\ 20/9 \\ 23/9 \\ 0 \end{bmatrix}$$

$$\vec{y}_2 = \vec{y}^{\perp} = \vec{y} - \operatorname{proj}_V \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 13/9 \\ 20/9 \\ 23/9 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/9 \\ -2/9 \\ 4/9 \\ 0 \end{bmatrix}$$

(3). Find a basis for $V^{\perp}$

$V^\perp = \ker\left(\begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{bmatrix}\right)$

$\begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$ $\begin{cases} x_1 + x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \end{cases}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -\frac{1}{2}x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, So $\left\{ \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $V^\perp$.