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§3.4 Coordinates (Homework: 1, 2, 6, 7, 19, 20, 28, 37–40, 47)

Theorem. [Unique Representation Theorem]

Let V be a subspace of $\mathbb{R}^n$ and let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for V. Then **each** vector $\vec{v}$ in V can be written as a linear combination

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p,$$

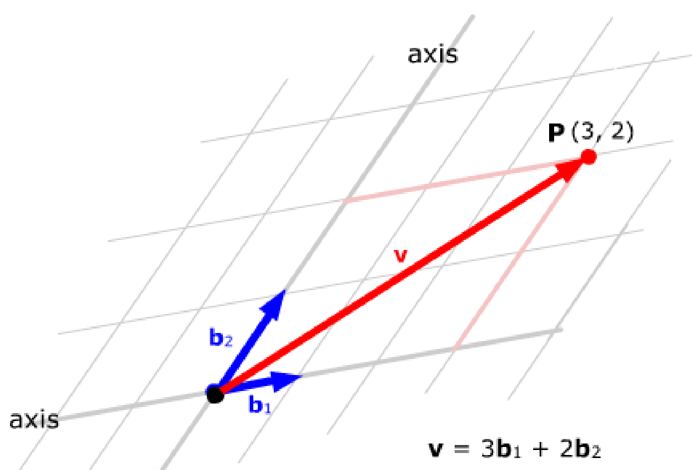
for a ***unique*** set of scalars $c_1, \dots, c_p$.

Definition. [Coordinates Relative to a Basis]

The **coordinates of** $\vec{v} \in V$ relative to $\mathscr{B}$ are the unique weights $c_1, \dots, c_p$ for which

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p,$$

In this case, we write $[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$



$$[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Example 1 (The standard basis for $\mathbb{R}^n$)**.**

The **standard basis** for $\mathbb{R}^n$ is the set $E = \{\vec{e}_1, \dots, \vec{e}_n\}$. The associated E-coordinates are called the **standard coordinates** of a vector in $\mathbb{R}^n$, and

$$[\vec{x}]_E = \vec{x}$$

Example 2 (Coordinates Relative to a Basis). Consider a basis $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Suppose $\vec{x} \in \mathbb{R}^2$ has the coordinate vector $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Find $\vec{x}$.

$$\vec{x} = -3\vec{b}_1 + 2\vec{b}_2 = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

$$= [\vec{b}_1 \ \vec{b}_2][\vec{x}]_B$$

Example 3 (The Change of Coordinates Matrix). Let $\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathscr{B}}$ of $\vec{x}$ relative to the basis $\mathscr{B}$ for $\mathbb{R}^2$ as in the above example.

$$c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{x}$$

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 2 & 8 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 3 & 6 \end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$c_1 = 4$
$c_2 = 2$

So $[\vec{x}]_B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Theorem.

Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$ be any vector. Let $P_{\mathscr{B}}$ be the $n \times n$ matrix whose columns are $\vec{b}_1, \cdots, \vec{b}_n$ written in the standard basis for $\mathbb{R}^n$

$$P_{\mathscr{B}} = \begin{bmatrix} \vec{b}_1 & \ldots & \vec{b}_n \end{bmatrix}$$

Then the standard coordinates of $\vec{x} \in \mathbb{R}$ can be calculated from the $\mathscr{B}$-coordinates $[\vec{x}]_{\mathscr{B}}$ of $\vec{x}$ as

$$\vec{x} = P_{\mathscr{B}} \cdot [\vec{x}]_{\mathscr{B}}.$$

Definition. [Change-of-coordinates Matrix]

The matrix $P_{\mathscr{B}}$ from the previous theorem is called the **change-of-coordinates matrix** from the basis $\mathscr{B}$ to the standard basis $\mathscr{E} = \{e_1, \dots, e_n\}$.

The change-of-coordinates matrix $P_{\mathscr{B}}$ is always **invertible**, and equation $\vec{x} = P_{\mathscr{B}} \cdot [\vec{x}]_{\mathscr{B}}$ can be used to find the $\mathscr{B}$-coordinates of $\vec{x}$ in terms of the standard coordinates of $\vec{x}$ as

$$[\vec{x}]_{\mathscr{B}} = P_{\mathscr{B}}^{-1} \cdot \vec{x}.$$

Example 4. Example3

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$[\vec{x}]_{B} = P^{-1}\vec{x} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Theorem. The matrix of a linear transformation

Let $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. There is an $n \times n$ matrix C such that

$$[T(\vec{x})]_{\mathscr{B}} = C[\vec{x}]_{\mathscr{B}}$$

The matrix C can by calculated by

$$C = \Big[[T(\vec{b}_1)]_{\mathscr{B}} \ [T(\vec{b}_2)]_{\mathscr{B}} \ \cdots \ [T(\vec{b}_n)]_{\mathscr{B}} \Big]$$

The matrix C is called the *matrix of T respect to basis $\mathscr{B}$, or $\mathscr{B}$-matrix.*

Suppose $T(\vec{x}) = A\vec{x}$. Denote $P = [\vec{b}_1 \ \vec{b}_2 \dots \vec{b}_n]$

Theorem.

$$A = PCP^{-1}$$

So

$$C = P^{-1}AP$$

Proof.

$$[A\vec{x}]_B = C \cdot [\vec{x}]_B$$

$$P^{-1}A\vec{x} = C\,P^{-1}\vec{x}$$

$$P^{-1}A = C P^{-1}$$

$$A = PCP^{-1}$$

If A and C satisfy $A = PCP^{-1}$, then A and C are called **similar**.

Example 5. Consider a basis $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Suppose a transformation T is defined by matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the matrix C of the transformation T respect to basis $\mathscr{B}$?

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C = P^{-1}AP = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

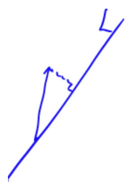
Example 6. Let T be the projection transformation onto a line $L = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$.

Find a basis $\mathscr{B}$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is diagonal.

we need a basis $B = \{\vec{b}_1\ \vec{b}_2\ \vec{b}_3\}$ such that

$$C = \left[[T(\vec{b}_1)]_B\ [T(\vec{b}_2)]_B\ [T(\vec{b}_3)]_B \right] = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

So $[T(\vec{b}_1)]_B = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$



So $T(\vec{b}_1) = c_1(\vec{b}_1)$, Similarly $T(\vec{b}_2) = c(\vec{b}_2)$

$T(\vec{b}_3) = c(\vec{b}_3)$

Choose $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ So $T(\vec{b}_1) = \vec{b}_1$

Choose $\vec{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, So $T(\vec{b}_2) = 0\,\vec{b}_2$

Chose $\vec{b}_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ So $T(\vec{b}_3) = 0\,\vec{b}_3$

$\vec{b}_1\ \vec{b}_2\ \vec{b}_3$ independent $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Definition. (Abstract) Vector space

A **vector space** is any nonempty set V of objects, called *vectors*, on which there are defined two **closed** operations,
- **vector addition (sum)**, and
- **multiplication by a scalar (scalar product)**,

subject to the rules below, called **axioms of a vector space**:
1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.
3. There is a zero vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$.
4. For each $\vec{u} \in V$, there is a vector $-\vec{u} \in V$ such that $\vec{u} + (-\vec{u}) = 0$.
5. $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$.
6. $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{v}$.
7. $c(d \cdot \vec{u}) = (cd)\vec{u}$.
8. $1 \cdot \vec{u} = \vec{u}$.

These must hold for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $c, d \in \mathbb{R}$.

In this definition, we use real numbers as scalar. The vector space is called real vector space. The axioms of a vector space imply that for all $\vec{u} \in V$, $c \in \mathbb{R}$,

$$0 \cdot \vec{u} = \vec{0}, \qquad c \cdot \vec{0} = \vec{0}, \qquad -\vec{u} = (-1)\vec{u}.$$

1. $\mathbb{R}^n$ is a vector space.

2. The set of all transformations $\mathbb{R}^m \to \mathbb{R}^n$ is a vector space.

2. The set of all $m \times n$ matrices is a vector space.

3. The set of all functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space.

Any definitions or theorems in Chapter 3 defined by **vectors** are true for the abstract vector spaces.

Like, **linear independence**, **subspace**, **linear combination**, **basis**, **dimension**.

1. Let P be the set of all polynomials.

$P = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\}$ for all real numbers $a_0\ a_1\ a_2 \cdots$

2. Let P_n be the set of all polynomials of degree $\leq n$.

$P = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$ for all real numbers $a_0\ a_1 \cdots a_n$

3. Let H be the set of all polynomials of degree exactly 3, with real coefficients.

No.

$H = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{l} a_0, a_1, a_2, a_3 \in \mathbb{R} \\ a_3 \neq 0 \end{array} \right\}$

4. Let $H = \{ax^4 + b \mid a, b \in \mathbb{R}\}$. Is H a subspace of P_4?

Yes.

5. Let $H = \{x^2 + a \mid a \in \mathbb{R}\}$. Is H a subspace of P?

No.

6. The set $U_{n\times n}$ of all $n \times n$ upper triangular matrices with real entries.

Yes.

7. The set $L_{n\times n}$ of all $n \times n$ lower triangular matrices with real entries.

Yes.

8. The set $D_{n\times n}$ of all $n \times n$ diagonal matrices with real entries.

Yes.

9. The set $T_{m\times n}$ of all $n \times n$ triangular matrices with real entries.

No

Definition.

Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. The map

$$T\colon V \to \mathbb{R}^n, \quad \text{given by} \quad T(\vec{x}) = [\vec{x}]_{\mathscr{B}}$$

is called the **coordinate mapping** from V to $\mathbb{R}^n$ with respect to $\mathscr{B}$.

The coordinate mapping allows us to view vectors $\vec{x}$ in the abstract vector space V by means of coordinates of vectors in the concrete and familiar vector space $\mathbb{R}^n$.

Theorem.

For any choice of basis $\mathscr{B}$ of the vector space V, the associated coordinate mapping $T(\vec{x}) = [\vec{x}]_{\mathscr{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$

Example 7 (The Coordinate Mapping)**.** Let V be the vector space of all polynomials of degree ≤ 2.

$T: V \longrightarrow \mathbb{R}^3$

$a_0 + a_1 x + a_2 x^2 \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$