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§3.3 The Dimension of a Subspace of $\mathbb{R}^n$

For a subspace V of $\mathbb{R}^n$, it has many different bases. However, they contain some common properties.

From §3.2, we know that: If the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is independent in V, and the set $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ spans V, then $m \geq p$.

Theorem.

If $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ and $\mathscr{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ are two bases for V, then $p = m$.

Definition. [The Dimension of a Vector Space]

The **dimension** of a vector space V is defined as

$$\dim V = \text{The cardinality of any basis for } V,$$

i.e., the number of elements in a basis.

By convention, the dimension of the zero vector space $V = \{\vec{0}\}$ is 0.

Example 1. The dimension of $\mathbb{R}^n$ is n, since $\mathbb{R}^n$ has a standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$.

Theorem. [The Basis Theorem]

Let V be a vector space with $\dim(V) = p \geq 1$.
• Any linearly independent set of exactly p elements in V is automatically a basis for V.
• Any set of p elements in V that span V, is automatically a basis for V.

Example 2. Is $\left\{ \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Yes. Since $\dim \mathbb{R}^3 = 3$ and those 3 vectors are independent, so by Basis Theorem, they form a basis for $\mathbb{R}^3$.

Example 3. Is $\left\{ \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No. There are more vectors than dimension.

Example 4. Is $\left\{ \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^4$?

No. There are fewer vectors than dimension.

Example 5. Is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No. $\operatorname{rank}(A) = 2$, the set is dependent.

Theorem. [The Dimensions of $\ker(A)$ and $\operatorname{im}(A)$]

Let A be an $n \times p$ matrix.
- The dimension of $\ker(A)$ is the number of free variables in the equation $A\vec{x} = \vec{0}$.
- The dimension of $\operatorname{im}(A)$ is the number of pivot columns in A.
- $\dim(\operatorname{im}(A)) = \operatorname{rank}(A)$.
- We always have the following

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = p.$$

Example 6. (Example 6 in §3.2) Find bases for the kernel and image of the transformation defined by $A = \begin{bmatrix} 0 & 0 & 2 & -8 & -1 \\ 1 & 6 & 2 & -5 & -2 \\ 2 & 12 & 2 & -2 & -3 \\ 1 & 6 & 0 & 3 & -2 \end{bmatrix}$.

Im(A) has a basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \\ -2 \end{bmatrix} \right\}$

rref $A = \begin{bmatrix} 1 & 6 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Ker(A) has basis $\left\{ \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\}$

So $\dim(\operatorname{Im}(A)) = 3$ $\quad \dim(\ker A) = 2$

Proposition.

Let A be an $n \times n$ square matrix.
A is **invertible**,
if and only if the columns vectors span $\mathbb{R}^n$;
if and only if the columns vectors of A are independent;
if and only if the columns vectors of A form a basis for $\mathbb{R}^n$.

Example 7. Can you find a 3×3 matrix A such that $\dim(\ker A) = \dim(\operatorname{im}(A))$?

No. $\dim(\ker A) + \dim(\operatorname{im}(A)) = 3$, so $\dim(\ker A) = \dim(\operatorname{im}(A)) = 1.5$. This is impossible.

Example 8. Can you find a 4×4 matrix A such that $\dim(\ker A) = \dim(\operatorname{im}(A))$?

Yes. For example $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Question: Can you find a 4×4 matrix A such that $\ker A = \operatorname{im}(A)$?

Example 9. If an 4×4 matrix $A = BC$ such that B is a 4×3 matrix and C is a 3×4 matrix. Is A invertible?

HW.39. No. Since $\operatorname{rank}(A) = \operatorname{rank}(BC) \leq \operatorname{rank} B \leq 3$. So, A is not invertible.
Or, we can think about the transformation of $A = BC$ is $\mathbb{R}^4 \xrightarrow{C} \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^4$. The image is not $\mathbb{R}^4$, so A is not invertible.

Example 10. A subspace V of $\mathbb{R}^n$ is called a **hyperplane** if V is defined by

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0$$

where at least one c_i is not zero. What is the dimension of V?

$A = [c_1 \ c_2 \ \cdots \ c_n]$ has $\operatorname{rank} A = 1$.
So $V = \ker A$ has dimension $n - 1$.

Example 11. Let T be the transformation defined by $A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$.

Suppose we already know $\textbf{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q1. Find **bases** for the **kernel** and **image** of T.

$x_1 - 2x_2 - x_4 = 0$

$x_3 + x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

A basis for $\ker(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

A basis for $\operatorname{im}(A)$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

Q2. What are the dimensions of for the **kernel** and **image** of A?

$\dim(\ker A) = 2; \dim(\operatorname{im} A) = 2;$

Q3. Is $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ in the kernel $\ker(A)$?

Compute $A\vec{u} = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Yes

Q4. Is $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the column of A?

$$[A\ \vec{v}] = \left[\begin{array}{cccc|c} -3 & 6 & -1 & 1 & 1 \\ 1 & -2 & 2 & 3 & 2 \\ 2 & -4 & 5 & 8 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 2 \\ -3 & 6 & -1 & 1 & 1 \\ 2 & -4 & 5 & 8 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+3R_1} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 2 \\ 0 & 0 & 5 & 10 & 7 \\ 0 & 0 & 1 & 2 & -3 \end{array}\right]$$

$$\xrightarrow{R_2-5R_3} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 22 \\ 0 & 0 & 1 & 2 & -3 \end{array}\right]$$

no solution.

Is $\vec{w} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ in the column of A?

Yes.