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§3.2 Bases and Linear Independence

1. Linear Independent sets

Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_t$ be vectors in $\mathbb{R}^n$. Then $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_t)$ is a subspace of $\mathbb{R}^n$.

The other direction is also correct:

> Any subspace V of $\mathbb{R}^n$ can be written as span of some vectors in V.

We can always write $V = \text{Span}(V)$. However, we want to write V as span of the smallest finite set.

For example, we can write image and kernel of a linear transformation as span of some vectors.

Example 1. (From §3.1) $A = \begin{bmatrix} 0 & 0 & 2 & 8 \\ 1 & 5 & 2 & -5 \\ 2 & 10 & 6 & -2 \end{bmatrix} = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$

The **image** of A is $\text{im}(A) = \text{Span}(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -2 \end{bmatrix}) \subset \mathbb{R}^3$.

The **kernel** is $\ker(A) = \text{Span}(\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \end{bmatrix}) \subset \mathbb{R}^4$.

In the image $\text{im}(A)$, we can see that $\vec{a}_2 = 5\vec{a}_1$. So, one of them is **redundant**.

$\text{im}(A) = \text{Span}(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -2 \end{bmatrix})$. Can we make it better?

Definition.

• The set of vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ is said to be **(linearly) independent** if the homogeneous vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

only has the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

• If there exists a nontrivial solution $(a_1, a_2, \dots, a_p)$, then $\vec{v}_1, \dots, \vec{v}_p$ is said to be **(linearly) dependent**. In this case,

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0}$$

is a **nontrivial relation** among the vectors $\vec{v}_1, \dots, \vec{v}_p$.

The vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$ is equivalent to using the matrix equation $A\vec{x} = \vec{0}$ or the augmented matrix $[A \mid \vec{0}]$.

The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is <u>independent</u>
if and only if the homogeneous equation <u>only has zero solution</u>;
if and only if there is <u>no free variable</u>;
if and only if <u>all columns contain pivots</u>;
if and only if $\underline{\text{rank}(A) = p}$;
if and only if $\underline{\ker(A) = \{\vec{0}\}}$.

From definition of dependence, we can obtain the following two properties for one or two vectors.

1. A set $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}$.
2. A set $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if one of the two vectors is a scalar multiple of the other vector.

We say a vector $\vec{v}_i$ is **redundant** if it is a linear combination of the preceding vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}\}$

The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is <u>independent</u> if and only if none of them is redundant.

The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is <u>dependent</u> if and only if at least one of them is redundant.

Proposition.

• If the set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of vectors contains the zero vector $\vec{0}$, then it is linearly dependent.
• If a subset of the set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is dependent.

Proposition.

If $p > n$, then a set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent.

Reason: If $p > n$, then, rank of $A = [\vec{v}_1 \ \dots \ \vec{v}_p]$ can not equal to p.

Warning: The preceding property does ***not*** say that $p \le n$ implies that $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent.

Example 2. (Checking Linear (in)dependence of Vectors)

(1.) $\left\{ \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \end{bmatrix} \right\}$ (2.) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \end{bmatrix} \right\}$

(3.) $\left\{ \begin{bmatrix} \sqrt{3} \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1.1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} \pi \\ 6 \\ 8 \end{bmatrix} \right\}$

(4.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \right\}$ (5.) $\left\{ \begin{bmatrix} 5 \\ 11 \\ 23 \\ 3.9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} \pi \\ 3 \\ 5 \\ 1.1 \end{bmatrix} \right\}$

(6.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (7.) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 8 \end{bmatrix} \right\}$

Dependent sets: (2), (3), (4), (5)
Independent sets: (1), (6), (7)

Example 3. (Linear (In)Dependence of Vectors)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}.$$

To determine whether or not the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linear dependent, we need to solve the vector equation:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

Equivalently, we need to solve the homogeneous equation $A\vec{x} = 0$ for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ 4 & 5 & 6 \end{bmatrix}.$$

We find the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ leading to } \begin{cases} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{cases}.$$

This shows that x_3 is a free variable, and $A\vec{x} = 0$ has nontrivial solutions, for example $(1, -2, 1)$. Hence, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linear dependent. A linear dependence relation is $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = 0$.
Accordingly, the columns of A are *not* linearly independent.

2. Basis of a subspace

Definition.

Let V be subspace of $\mathbb{R}^n$. A subset $B = \{\vec{b}_1, \dots, \vec{b}_p\}$ of V is called a **basis** for V if

(i) B is linearly independent, and

(ii) $\text{Span}\{\vec{b}_1, \dots, \vec{b}_p\} = V$.

Example 4. Standard bases for $\mathbb{R}^n$

$\mathbb{R}^2$ has a standard basis $\left\{\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\mathbb{R}^3$ has a standard basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

A subset B of a vector space V has a "greater chance" of being
- linearly independent, if it has fewer vectors;
- a spanning set of V, if it has more vectors.

More precisely, we have

Theorem.

If the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is independent in V, and the set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ spans V, then $m \geq p$.

$[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_p] = [\vec{w}_1\ \vec{w}_2 \cdots \vec{w}_m]\, C \Leftarrow$ $\vec{v}_i = c_{1i}\vec{w}_1 + c_{2i}\vec{w}_2 + \cdots + c_{mi}\vec{w}_m$

$m \times p$ matrix C. $= (\vec{w}_1\ \vec{w}_2 \cdots \vec{w}_m)\begin{bmatrix} c_{1i} \\ \vdots \\ c_{mi} \end{bmatrix}$

$V = WC$

$WC\vec{x} = \vec{0}$ only has zero solution

$\Rightarrow C\vec{x} = \vec{0}$ only has zero solution

$\Rightarrow$ no free variable $\Rightarrow m \geq p$

Example 5. Find a basis for the subspace im(A) and ker(A) in Example 1.

① $\ker(A) = \text{Span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ These two vectors are independent. So they form a basis.

② $\text{Im}(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \\ -2 \end{bmatrix} \right\}$

$\vec{u}_3 = 4\vec{u}_2 - 13\vec{u}_1$

So $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\}$ is a basis for $\text{Im}(A)$.

Let A be an $n \times p$ matrix. The following theorems give us a method for finding bases for the subspaces im(A) and ker(A).

Theorem. [Basis for ker(A)]

Solve the matrix equation $A\vec{x} = \vec{0}$. Write $\vec{x}$ as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_p$ with the weights corresponding to the free variables.
Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for ker(A).

Theorem. [Basis for im(A)]

A basis for the image im(A) is given by the pivot columns of A.

Example 6. Find bases for the kernel and image of the transformation defined by $A = \begin{bmatrix} 0 & 0 & 2 & -8 & -1 \\ 1 & 6 & 2 & -5 & -2 \\ 2 & 12 & 2 & -2 & -3 \\ 1 & 6 & 0 & 3 & -2 \end{bmatrix}$.

From §1.2, we already know $\textbf{rref}(A) = \begin{bmatrix} 1 & 6 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Im(A) has a basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \\ -2 \end{bmatrix} \right\}$

$x_1 + 6x_2 + 3x_4 = 0$
$x_3 - 4x_4 = 0$
$x_5 = 0$

$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_3 = 4x_4 \\ x_5 = 0 \\ x_2, x_4 \text{ free} \end{cases}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_2 - 3x_4 \\ x_2 \\ 4x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$

So ker(A) has a basis $\left\{ \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\}$

A vector $\vec{b} \in \mathbb{R}^n$ belongs to the column space of A if and only if there exist numbers $x_1, \ldots, x_p$ such that

$$x_1\vec{a}_1 + \cdots + x_p\vec{a}_p = \vec{b}.$$

This in turn happens if and only if the matrix equation $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$.

This last point shows that im(A) $= \mathbb{R}^p$ if and only if the matrix equation $A\vec{x} = \vec{b}$ has a solution $\vec{x}$ for every choice of $\vec{b} \in \mathbb{R}^n$.

3. Coordinates

Theorem. [Unique Representation Theorem]

Let V be a subspace of $\mathbb{R}^n$ and let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for V. Then **each** vector $\vec{v}$ in V can be written as a linear combination

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p,$$

for a ***unique*** set of scalars $c_1, \ldots, c_p$.

Said differently, if

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p, \text{ and } \vec{v} = d_1 \cdot \vec{b}_1 + \cdots + d_p \cdot \vec{b}_p$$

then $c_1 = d_1$, $c_2 = d_2$, $\ldots$, $c_p = d_p$.

Definition. [Coordinates Relative to a Basis]

Let V be a subspace of $\mathbb{R}^n$ and let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for V. The **coordinates of** $\vec{v} \in V$ relative to $\mathscr{B}$ are the unique weights $c_1, \ldots, c_p$ for which

$$\vec{v} = c_1 \cdot \vec{b}_1 + \cdots + c_p \cdot \vec{b}_p,$$

In this case, we write

$$[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

More details about coordinates in §3.4.