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§3.1 Image and Kernel of a Linear Transformation

Let T be a (linear) transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Definition. (Image and Kernel)

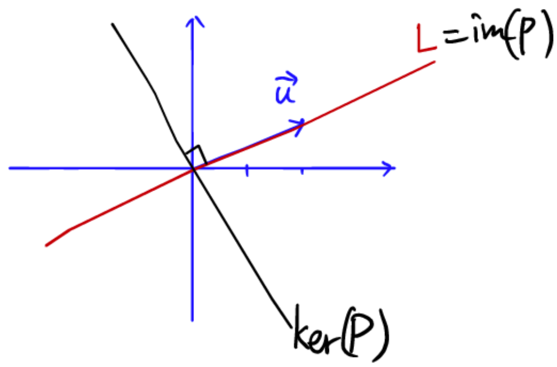
The **image** of T is defined as

$$\text{im}(T) := \{T(\vec{x}) \mid \text{ all } \vec{x} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

The **kernel** of T is defined as

$$\ker(T) := \{\vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0}\} \subset \mathbb{R}^m$$

Example 1. Let L be the line spanned by $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the image and kernel of P defined by the orthogonal projection onto L.



More precisely,

$$P = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\text{Im}(P) = \text{Span}\left\{ \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$\ker(P) = \{ \text{solutions of } 2x_1 + x_2 = 0 \}$$

Example 2. (H.W.)Think about the reflection and rotation on $\mathbb{R}^2$

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Denote R: Reflection. $\text{im}(R) = \mathbb{R}^2$ and $\ker(R) = \{\vec{0}\}$
O: rotation. $\text{im}(O) = \mathbb{R}^2$ and $\ker(O) = \{\vec{0}\}$

We know that any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is defined by a $n \times m$ matrix $A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_m]$.

The image of T is

$$\begin{aligned}\operatorname{im}(T) &= \{A(\vec{x}) \mid \text{ all } \vec{x} \in \mathbb{R}^m\} \\ &= \{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m \mid \text{ all real numbers } x_i\} \\ &= \{\text{all linear combinations of } \vec{a}_1, \vec{a}_2, \cdots, \vec{a}_m\} \\ &= \operatorname{Span}(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_m)\end{aligned}$$

The kernel of T is the solution set

$$\begin{aligned}\ker(T) &= \{\vec{x} \in \mathbb{R}^n \mid A(\vec{x}) = \vec{0}\} \\ &= \{\text{all solutions of } A(\vec{x}) = \vec{0}\}\end{aligned}$$

Example 3. Find a transformation which has kernel as a plane

$$2x - y + 3z = 0.$$

How many such transformation can we find?

Plane={all solutions of $2x - y + 3z = 0$} ={$vx \in \mathbb{R}^3 | A\vec{x} = \vec{0}$} where $A = [2 \quad -1 \quad 3]$.

$[2 \ -1 \ 3]$ defines such a transformation. $\mathbb{R}^3 \to \mathbb{R}^1$

$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$ defines such a transformation $\mathbb{R}^3 \to \mathbb{R}^2$

Example 4. What is the geometry of image and kernel of the transformation defined by $T(\vec{x}) = \vec{v} \cdot \vec{x}$ for $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$?

$T(\vec{x}) = \vec{v} \cdot \vec{x}$

$= 2x_1 - x_2 + 3x_3 \quad : \mathbb{R}^3 \to \mathbb{R}^1$

- matrix of T is $[2 \ -1 \ 3]$
- $\operatorname{im}(T) = \mathbb{R}^1$
- $\ker(T) = \{$ all vectors $\vec{x} \mid \vec{v} \cdot \vec{x} = 0\} = \{$all vectors orthogonal to $\vec{v}\}$

Example 5. Find the image and kernel of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by matrix $A = \begin{bmatrix} 0 & 0 & 2 & 8 \\ 1 & 5 & 2 & -5 \\ 2 & 10 & 6 & -2 \end{bmatrix} = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$

The **image** of T is $\text{im}(T) = \text{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$.

We know that the kernel of T is $\ker(T) = \{\text{all solutions of } A\vec{x} = 0\}$.

We calculated that $\textbf{rref}(A) = \begin{bmatrix} 1 & 5 & 0 & -13 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solutions of $A(\vec{x}) = \vec{0}$ can be described as vector form:

$$\vec{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \end{bmatrix} = x_2\vec{v}_1 + x_4\vec{v}_2$$

So the **kernel** is $\ker(T) = \text{Span}(v_1, v_2)$.

Consider an $n \times m$ matrix. We know that $\ker(T)$ is the set of all solutions of $A(\vec{x}) = \vec{0}$. We can get the following results.

Theorem.

(1.) $\ker(A) = \{\vec{0}\}$ if and only if $A(\vec{x}) = \vec{0}$ only has zero solution.

(2.) $\ker(A) = \{\vec{0}\}$ if and only if $\text{rank}(A) = m$.

(3.) If $\ker(A) = \{\vec{0}\}$, then $m \leq n$.

Recall that A is invertible if and only if $\text{rank}\, A = n$. Then,

Theorem.

Let A be an $n \times n$ square matrix.
(1.) A is invertible if and only if $\ker(A) = \{\vec{0}\}$.
(2.) A is invertible if and only if if and only if $\text{im}(A) = \mathbb{R}^n$.

Definition. (Subspace)

A **subspace** of the vector space $\mathbb{R}^n$ is a subset H of $\mathbb{R}^n$ that satisfies the following three properties.
(1). $\vec{0} \in H$.
(2). If $\vec{u}, \vec{v} \in H$ then $\vec{u} + \vec{v} \in H$. (Closed under addition)
(3). If $\vec{u}, \vec{v} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$. (Closed under scalar product)

(1) $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$, called **zero space**.
(2) Any line or plane passing zero in $\mathbb{R}^n$ is a subspace.

Theorem.

Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_t$ be vectors in $\mathbb{R}^n$. Then $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_t)$ is a subspace of $\mathbb{R}^n$.

Proof (for t=3)

$S = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \{ x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 \mid \text{any } x_1, x_2, x_3 \in \mathbb{R} \}$

① $\vec{0} \in S$

② For any $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$ and $y_1\vec{u}_1 + y_2\vec{u}_2 + y_3\vec{u}_3 \in S$,

the sum is $(x_1+y_1)\vec{u}_1 + (x_2+y_2)\vec{u}_2 + (x_3+y_3)\vec{u}_3$ is in S.

(closed under sum)

③ For any $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 \in S$ and $c \in \mathbb{R}$, we have

$c(x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3) = (cx_1)\vec{u}_1 + (cx_2)\vec{u}_2 + (cx_3)\vec{u}_3$ is in S

(closed under scalar product)

So S is a subspace.

Theorem.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

1. $\text{im}(T)$ is a subspace of $\mathbb{R}^n$.
2. $\ker(T)$ is a subspace of $\mathbb{R}^m$.

Example 6. Determine which of the following set is a subspace (vector space).

1. Let L be the set of vectors on the line $2x_1 - x_2 = 0$.

Yes.

2. Let L be the set of vectors on the line $2x_1 - x_2 = 1$.

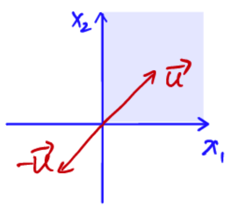
No. $\vec{0} \notin L$.

3. Let H be the set of vectors on the plane $3x_1 - 5x_2 + x_3 = 0$.

Yes.

4. Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}$.

No. If $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $-\vec{u} \notin V$.



5. The union of the first and second quadrants in the xy-plane: $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y \geq 0 \right\}$

No. If $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $-\vec{u} \notin V$.