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§2.4 The Inverse of a Linear Transformation

The Inverse of a matrix

Definition.

An $n \times n$ matrix A is called **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Remark: 1. Since the role of A and B in the definition is symmetric, if B is the inverse of A, then A is the inverse of B.
2. If A is invertible, then it has only one inverse. The inverse of A is denoted as A^{-1}.

Theorem. [Algorithm for Computing A^{-1}]

Given an $n \times n$ matrix A.
1. Define an $n \times 2n$ "augmented matrix "

$$[A \mid I_n]$$

2. Using elementary row operations to find **rref**$[A \mid I_n]$.
(i). If **rref**$[A \mid I_n] = [I_n \mid C]$, then $C = A^{-1}$.
(ii). If this is not possible, then A is not invertible.

Example 1. Find the inverse of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] = [I \mid A^{-1}]$$

So $A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$

Theorem. [Invertibility of 2×2 Matrices]

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.
The formula for the inverse matrix of A is

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We call $ad - bc$ the **determinant** of A, denoted by $\det(A)$ or $|A|$.

Geometric meaning of determinant: The absolute value of the determinant of a 2×2 matrix A is the area of the parallelogram spanned by the column vectors $\vec{a}_1$ and $\vec{a}_2$.

Example 2.

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{16-15} \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

$\det(A) = 16 - 15 = 1$. $\vec{a}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$|\det(A)|$ is the area of the parallelogram spanned by the column vectors $\vec{a}_1$ and $\vec{a}_2$.

Example. The inverse of the elementary matrices.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_1 \qquad E_1^{-1} = E_1$$

$$\xrightarrow{3R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_2 \qquad E_2^{-1} = \begin{bmatrix} 1 & & & \\ & \frac{1}{3} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_3 \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3. The inverse of the elementary matrices.

$$E_{ij}^{-1} = E_{ij}, \quad E_i(c)^{-1} = E_i(1/c), \quad E_{ij}(d)^{-1} = E_{ij}(-d)$$

Theorem.

Let A and B be $n \times n$ matrices.

- If A is invertible, then so is A^{-1} and

$$(A^{-1})^{-1} = A.$$

- If A is invertible, then so is A^T and

$$(A^T)^{-1} = (A^{-1})^T.$$

- If A and B are invertible, then so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}$$

- If A is invertible and $k \neq 0$, then so is kA and

$$(kA)^{-1} = \frac{1}{k}A^{-1}.$$

- Suppose A is invertible. If $AB = AC$, or $BA = CA$, then $B = C$.

Each one can be verified using definition. We use the product as an example, $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ by associativity of products. Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = BIB^{-1} = BB^{-1} = I$. So AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Example 4. If A and B are $n \times n$ invertible matrices, is $A + B$ invertible?

No. For example, if $B = -A = I_2$, then $A + B = 0$ is not invertible.

Example 5. If A is invertible, is A^m invertible? What is the inverse?

Yes. $(A^m)^{-1} = (A^{-1})^m$, since $A^m(A^{-1})^m = A \cdots A \cdot A^{-1} \cdots A^{-1} = I$.

Theorem. [The inverse matrix theorem]

Let A be an $n \times n$ matrix. Then the next statements are all equivalent (that is, they are either all true or all false).
(1) The matrix A is invertible.
(2) There is a square matrix B such that $BA = I$.
(3) The linear system $A\vec{x} = \vec{0}$ has only the trivial solution.
(4) $\operatorname{rank} A = n$.
(5) The reduced row echelon form of A is identity matrix, i.e. $\mathbf{rref}(A) = I_n$.
(6) The matrix A is a product of elementary matrices.
(7) There is a square matrix C such that $AC = I_n$.
(8) The linear system $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
(9) The linear transformation T_A is one-to-one.
(10) The linear transformation T_A is onto.

Proof.
(1)⇒(2) Obvious.
(2)⇒(3) Multiply B on left, we get $BA\vec{x} = \vec{0}$. So $\vec{x} = \vec{0}$ is the only solution.
(3)⇒(4) By Proposition in §1.3.
(4)⇒(5) Obvious.
(5)⇒(6) $EA = \mathbf{rref}(A) = I_n$ where $E = E_1 \cdots E_s$ is a product of elementary matrices. So, $A = E_s^{-1} \cdots E_1^{-1}$ is a product of elementary matrices.
(6)⇒(1) The reason is that elementary matrices are invertible matrices and product of invertible matrices are invertible.
We have proved that (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6)⇒(1). So, the first six statements are equivalent.
(1)⇒(7) is obvious. (7)⇒(4): $\operatorname{rank}(AC) = \operatorname{rank}(I_n) = n$. So $\operatorname{rank} A \geq \operatorname{rank}(AC) = n$. This shows that (7) is also equivalent to the above six statements.
(8)⇒(6) is obviously. (1)⇒(8). Since $A^{-1}A = I$ and A^{-1} is unique, then $\vec{x} = A^{-1}\vec{b}$ is the unique solution. Hence the first eight statements are equivalent.
T_A is one-to-one if and only if (3). Hence the first nine statements are equivalent.
Since $T_A(\vec{x}) = A\vec{x}$, (8)⇒(10) is clear. (10)⇒(7) is clear.

This is a standard method to show equivalent statements. We don't have to show the other directions. Some of them may hard to show directly.

Example 6. If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

Yes. $A^{-1} = BC$ and $C^{-1} = AB$.
From $C^{-1} = AB$, we have $I_n = CC^{-1} = CAB$, so $B^{-1} = CA$.

Proposition.

Suppose A and B are $n \times n$ matrices. If AB is invertible, then both A and B are invertible.

Since AB is invertible, then there is a matrix C such that $ABC = I_n$. Use above example we proved the result.

Another proof for the proposition is from $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

The Inverse of a Linear Transformation

Definition (Invertible Transformations)

A transformation $T\colon X \to Y$ is said to be **invertible** (or **bijective**) if the equation $T(\vec{x}) = \vec{y}$ has a unique solution $\vec{x}$ for any $\vec{y} \in Y$.
Equivalently, there is a transformation $S\colon Y \to X$ such that

$$S(T(\vec{x})) = \vec{x}, \quad \text{for all } \vec{x} \in X;$$

$$T(S(\vec{y})) = \vec{y}, \quad \text{for all } \vec{y} \in Y.$$

Remark. Such a transformation $S\colon Y \to X$, if it exists, is unique and is called the **inverse transformation** of T. We shall use the notion $S = T^{-1}$ to denote the inverse transformation of T.

Recall that for an $n \times n$ matrix A we defined the associated linear transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_A(\vec{x}) = A\vec{x}.$$

Definition (Invertible Linear Transformations)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **invertible** if for any $\vec{y} \in \mathbb{R}^n$ there is a unique $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

Suppose the inverse $(T_A)^{-1}$ is defined by matrix B, then $T_A(T_B) = \text{id}$ and $T_B(T_A) = \text{id}$. Equivalently,

$$AB = BA = I_n$$

In this case, we will denote the inverse of A as A^{-1}

Theorem. [Invertibility of Linear Transformations]

The linear transformation $T_A\colon \mathbb{R}^m \to \mathbb{R}^n$ is invertible if and only if $m = n$ and A is an invertible matrix. In this case, the inverse of T_A is the linear transformation

$$(T_A)^{-1} = T_{A^{-1}}.$$

MATLAB functions We list some basic MATLAB functions about linear system here.

Input matrices and vectors to MATLAB:

```
A=[1 -3 -5; 1 -1 -2; 3 -1 1];
B=[1 1 1; 2 3 2; 3 8 2];
b=[1; 0; 3];
c = [2 1 8];
```

Sum and scalar product

```
A+B
3A
```

transpose Transpose vector or matrix A^T

```
C= A.'
C = transpose(A)
```

mtimes Matrix multiplication AB

```
C = A*B
C = mtimes(A,B)
```

mpower Matrix power A^k

```
C = A^3
C = mpower(A,3)
```

inv Matrix inverse A^{-1}

```
B^(-1)
inv(B)
```

rank Rank of matrix A

```
rank(A)
```