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§2.3 Matrix Products

• We previously defined the multiplication of an $n \times m$ matrix A and an m-dimensional vector $\vec{x}$, which is itself a $m \times 1$ matrix. The result $A \cdot \vec{x}$ is an n-dimensional vector, which is the same as an $n \times 1$ matrix.

$$(n \times m \text{ matrix}) \cdot (m \times 1 \text{ matrix}) = n \times 1 \text{ matrix}.$$

We shall next generalize this to multiplying more general matrices.

Definition:

Let A be an $n \times m$ matrix and B be a $m \times p$ matrix with columns $\vec{b}_1, \ldots, \vec{b}_p$. We then define the ***product*** of A and B, to be the $n \times p$ matrix

$$AB := [A \cdot \vec{b}_1 \quad A \cdot \vec{b}_2 \quad \ldots \quad A \cdot \vec{b}_p]$$

If the number of columns of A does not equal the number of rows of B, **then** AB **is not defined**!

• **The Row-Column Rule for Computing** AB

Let A be an $n \times m$ matrix whose (i, j)-th entry is a_{ij}.

Let B be an $m \times p$ matrix whose (i, j)-th entry is b_{ij}.

Then the (i, j)-th entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj},$$

which equals the dot product of the i-th row of A with the j-th column of B.

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$$

Example Calculate AB for $A = \begin{bmatrix} -3 & 5 \\ 4 & 2 \\ 1 & -5 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}$.

$$AB=\begin{bmatrix}-3 & 5\\ 4 & 2\\ 1 & -5\end{bmatrix}\begin{bmatrix}2 & -4\\ -4 & 1\end{bmatrix}=\begin{bmatrix}(-3)2+5(-4) & (-3)(-4)+5\\ (4)2+2(-4) & 4(-4)+2\\ (1)2+(5)(-4) & (-4)-5\end{bmatrix}=\begin{bmatrix}-26 & 17\\ 0 & -14\\ 22 & -9\end{bmatrix}$$

♢ Properties of Matrix Multiplication

Theorem. (Properties of Matrix Multiplication)

Let A be an $n\times m$ matrix, and let B and C be matrices for which the indicated operations are defined. Let In denote the $n\times n$ identity matrix.

- $A(BC)=(AB)C$. (Associativity of matrix multiplication)
- $A(B+C)=AB+AC$. (Left Distributive Law)
- $(A+B)C=AC+BC$. (Right Distributive Law)
- $r(AB)=(rA)B$. where r is any scalar.
- $I_nA=A=AI_m$. (Identity Law for Matrix Multiplication)

Each one is proved by direct verification. Let us verify the first associativity property. The rest verifications are easy. Suppose B is a $n\times p$ matrix and C is a $p\times q$ matrix. We compare the (i,j) position of both sides, using the sum notation.
For any $1\le i\le m$ and $1\le j\le q$,

$$[A(BC)]_{ij}=\sum_{k=1}^{n}a_{ik}(BC)_{kj}=\sum_{k=1}^{n}a_{ik}\left(\sum_{l=1}^{p}b_{kl}c_{lj}\right)=\sum_{k=1}^{n}\sum_{l=1}^{p}a_{ik}b_{kl}c_{lj}$$

$$[(AB)C]_{ij}=\sum_{l=1}^{p}(AB)_{il}c_{lj}=\sum_{l=1}^{p}\left(\sum_{k=1}^{n}a_{ik}b_{kl}\right)c_{lj}=\sum_{l=1}^{p}\sum_{k=1}^{n}a_{ik}b_{kl}c_{lj}=\sum_{k=1}^{n}\sum_{l=1}^{p}a_{ik}b_{kl}c_{lj}$$

So, $A(BC)=(AB)C$.

Example 1. $AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ 3 & 7\end{bmatrix}$

$BA=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}4 & 6\\ 3 & 4\end{bmatrix}$

Example 2. $AB=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}3 & 4\\ 0 & 0\end{bmatrix}$

$AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

Example 3. $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Remark (Non-Properties of Matrix Multiplication)

Some familiar arithmetic properties of real numbers do **not** translate to analogue properties of matrices.

- Even when both AB and BA are defined, generally $AB \neq BA$.
- If $AB = AC$ it does not generally follow that $B = C$ (even when $A \neq 0$).
- If $AB = 0$, it does not generally follow that either A or B is the zero matrix.

Definition.

If A is an $n \times n$ matrix and $k \geq 1$ an integer, we define the k-th power of A, denoted by A^k, as

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A}_{k \text{ factors}}.$$

Example. Calculate X^2, X^3, X^4, ... for the following matrices

$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

$A^3 = A \quad A^4 = A \quad A^5 = A \quad \cdots\cdots$

$$B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$B^3 = B^2 B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$\vdots$

$$B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C^3 = C^2 C = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$C^n = 0$ for $n \geq 3$.

$$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 1^3 & & \\ & 2^3 & \\ & & 3^3 \end{bmatrix} \quad \cdots \quad D^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}$$

Example 4. (HW 18) Find all matrices commute with

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

Find all $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $BA = AB$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So $\begin{bmatrix} 2a-3b & 3a+2b \\ 2c-3d & 3c+2d \end{bmatrix} = \begin{bmatrix} 2a+3c & 2b+3d \\ -3a+2c & -3b+2d \end{bmatrix}$

So $2a-3b = 2a+3c \Rightarrow -b = c$

$3a+2b = 2b+3d \Rightarrow a = d$

$2c-3b = -3a+2c \Rightarrow a = d$

$3c+2d = -3b+2d \Rightarrow -b = c$

So All matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ commute with $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

$a, b \in \mathbb{R}$.

♢ Product of block matrices.

Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

Here, suppose all matrix products are well defined.

Block products can simply the computation if some blocks are zero matrices.

Example 5. $AB = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix}$

$$AB = [A_1\ A_2]\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

♢ Transpose of a matrix

Definition.

Given an $m \times n$ matrix A, we define the ***transpose matrix*** A^T, as the $n \times m$ matrix whose (i, j)-th entry is the (j, i)-th entry of A.
Said differently, the rows of A^T are the columns of A, and the columns of A^T are the rows of A.

Example 6. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

Example 7. The dot product can be written as matrix product

$$\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$$

Theorem. [Properties of Matrix Transposition]

Let A and B be matrices such that the indicated operations are well defined.
- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$.
- $(rA)^T = rA^T$ for any scalar r.
- $(AB)^T = B^T A^T$.

Theorem.

- $\text{rank}(A) = \text{rank}(A^T)$.

- If AB is defined, then $\text{rank}(AB) \le \text{rank}\, A$ and $\text{rank}(AB) \le \text{rank}\, B$.

Definition.

A square matrix A is called **symmetric** if $A^T = A$. The matrix A is called **skew-symmetric** if $A^T = -A$.

For example, the reflection matrix is symmetric and the rotation matrix is skew-symmetric.

♢ Elementary matrices

Definition. [Elementary matrices]

An $n \times n$ matrix E is called **elementary** if it is obtained from the identity matrix I_n by a *single* row operation.
E_{ij} denotes the elementary matrix obtained by switching the i-th and j-th rows of the identity matrix.
$E_i(c)$ denotes the elementary matrix obtained by multiplying the i-th row by the nonzero constant c.
$E_{ij}(d)$ denotes the elementary matrix adding d times the j-th row to the i-th row. (The order is from right to left)

Example 8. 3×3 matrices: $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{12}(3) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Proposition. [Elementary matrices multiplications]

Multiply a matrix A with an elementary on the **left** side is equivalent to an elementary row operation is performed on the matrix A.

Example 9. $E_{ij}A$ is the matrix obtained from A by switch the i-th row and the j-the row.

$$A \xrightarrow{R_i \leftrightarrow R_j} E_{ij}A$$

Example 10. $E_i(c)A$ is the matrix obtained from A by multiplying the i-th row by the nonzero constant c.

$$A \xrightarrow{cR_i} E_i(c)A$$

Example 11. $E_{ij}(d)A$ is the matrix obtained from A by adding d times the j-th row to the i-th row.

$$A \xrightarrow{R_i + dR_j} E_{ij}(d)A$$

Remark: Multiply a matrix A with an elementary on the right side is equivalent to an elementary column operation is performed on the matrix A.