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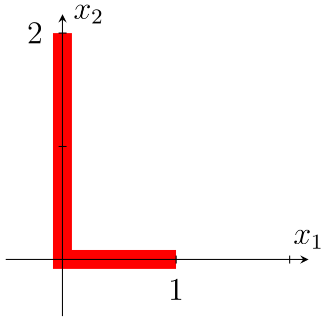
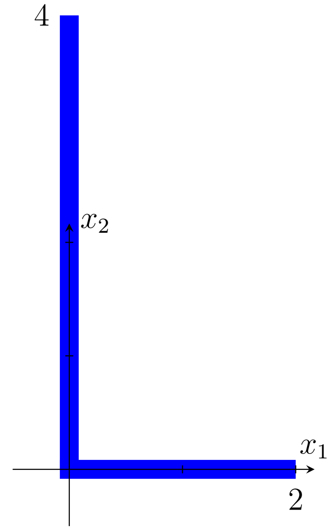
§2.2 Linear Transformation in Geometry

Recall that given an $n \times m$ matrix A there is a linear transformation defined by $T(\vec{x}) = A \cdot \vec{x}$. Let us look at the following **examples**.

Example 1.

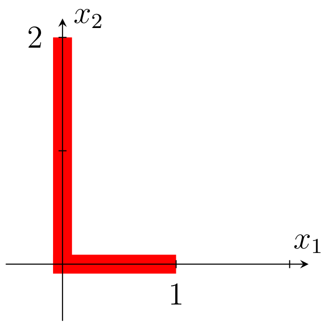
(1.) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (Scaling)

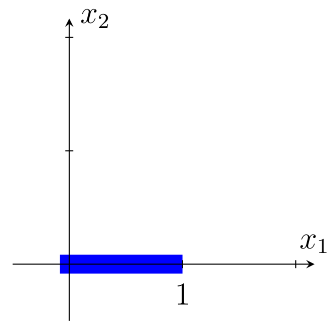




$$T_A(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

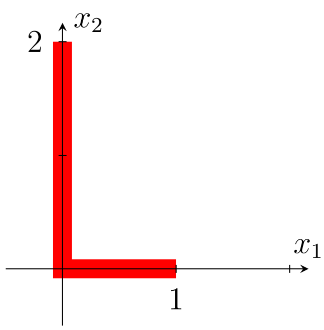
(2.) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (Orthogonal Projection on to x_1-axis)

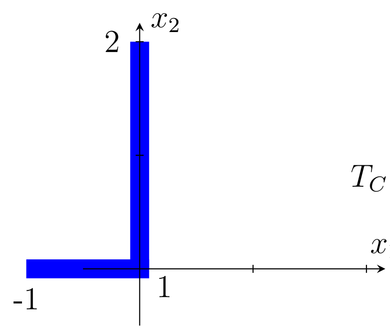




$$T_B(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

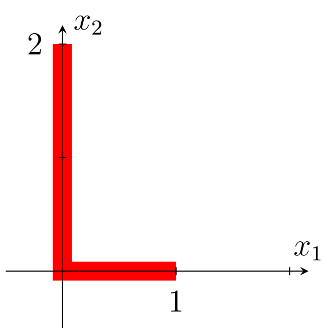
(3.) $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (Reflection about x_2-axis)

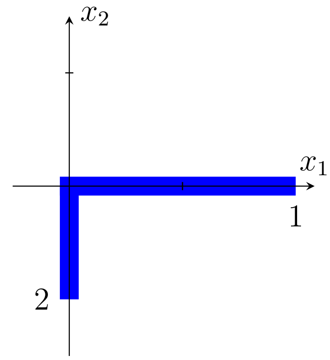




$$T_C(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

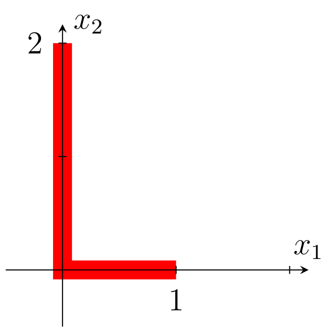
(4.) $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (Rotate 90° clockwise)

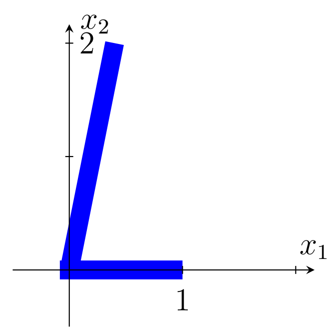




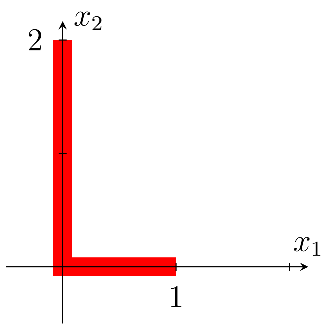
$$T_D(\vec{x}) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

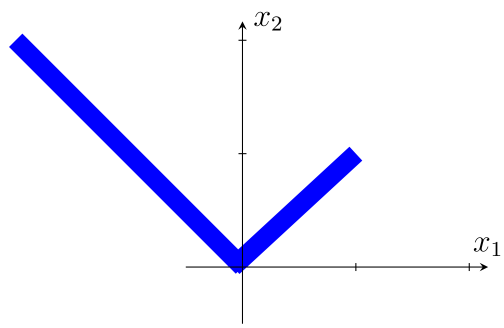
(5.) $E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ (Horizontal shear)





(6.) $F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (Rotate 45° and scaling by $\sqrt{2}$)





1. Scaling

For any constant $k > 0$, the matrix

$$kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

defines a **scaling** transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(1) If $k > 1$ this is a dilation (or enlargement).

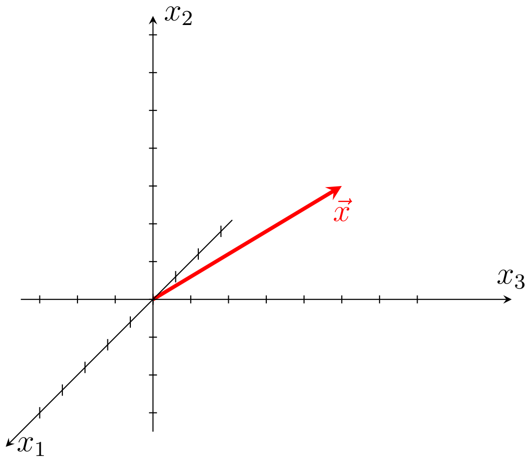
(2) If $0 < k < 1$, this is a contraction (or shrinking).

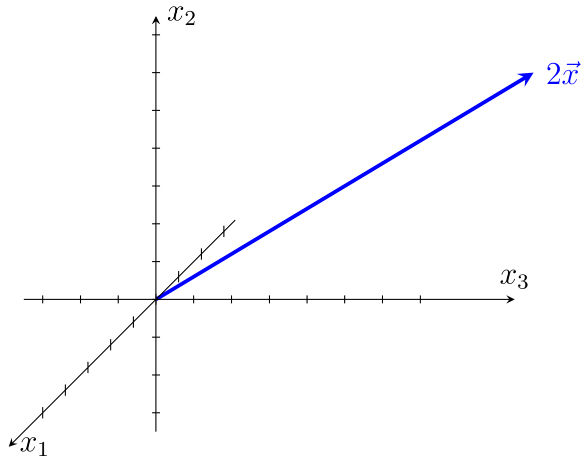
In general, for any constant $k > 0$, the matrix kI_n defines a scaling transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Moreover, the scaling transformation is given by $T(\vec{x}) = k\vec{x}$

For example

$$2I_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

defines transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = 2\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.





2. Orthogonal Projection

Recall that for vectors $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, the **dot product** of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Theorem. [Properties of the Inner Product]

For vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$, the following hold:

(1.) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

(2.) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.

(3.) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$.

(4.) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.

Definition. [Length of a Vector]

The **length** or **norm** of a vector $\vec{v} \in \mathbb{R}^n$, denoted by $||\vec{v}||$, is defined as

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, \ldots, v_n$ are the coordinates of $\vec{v}$.

Example 2. Find the length of the following vectors.

$$\vec{u} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$||\vec{u}|| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1.$

$||\vec{v}|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$

$||\vec{v}|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}.$

A vector $\vec{u}$ is called an **unit vector** if $||\vec{u}|| = 1$.

If a vector $\vec{w}$ is not an unit vector, we can find a unit vector on the same direction defined by

$$\frac{\vec{w}}{||\vec{w}||}$$

and called the **normalization** of $\vec{w}$.

Example 3. Find the normalization of the vectors in Example 2.

$\vec{u}$ is a unit vector

$$\frac{\vec{v}}{||\vec{v}||} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \qquad \frac{\vec{w}}{||\vec{w}||} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 3/\sqrt{30} \\ 4/\sqrt{30} \end{bmatrix}$$

Theorem.

For any vector $\vec{v} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$ one obtains

$$||c \cdot \vec{v}|| = |c| \cdot ||\vec{v}||.$$

Definition. (Angles Between Vectors)

The **angle between two nonzero vectors** $\vec{u}, \vec{v} \in \mathbb{R}^n$ is the the angle $0 \leq \theta \leq \pi$ satisfying

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \cos\theta.$$

Or we can write

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}.$$

In particular, when $\vec{u} \cdot \vec{v} = 0$, the angle $\theta = \frac{\pi}{2}$.

Definition. [Orthogonal Vectors]

Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are said to be **orthogonal** or **perpendicular** if $\vec{u} \cdot \vec{v} = 0$.

Example 4. Find the angle between the following pairs of vectors.

$$\vec{u} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \qquad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}$$

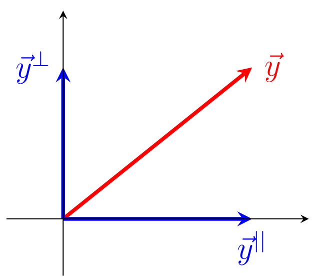
$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-1+2\sqrt{3}}{\sqrt{4}\sqrt{5}} = \frac{-1+2\sqrt{3}}{\sqrt{20}}$$

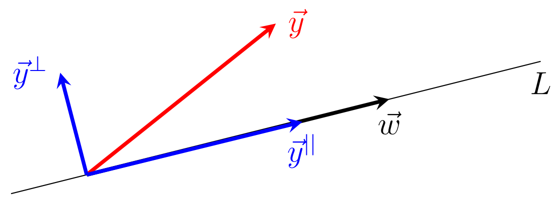
$$\theta = \arccos\left(\frac{-1+2\sqrt{3}}{\sqrt{20}}\right)$$

$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{0}{\|\vec{u}\|\|\vec{v}\|} = 0$$

$\vec{u}$ is perpendicular to $\vec{v}$

so $\theta = \frac{\pi}{2}$





Definition. (Orthogonal Projection Onto A Line)

Let $\vec{w}$ be a nonzero vector in $\mathbb{R}^n$ and let $L = \text{Span}\{\vec{w}\}$ be the line in $\mathbb{R}^n$ spanned by $\vec{w}$. For a given vector $\vec{y} \in \mathbb{R}^n$, the vector

$$y^{||} = \text{proj}_L(\vec{y}) := \left(\frac{\vec{y}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

is called the **orthogonal projection of** $\vec{y}$ **onto** L (or onto $\vec{w}$) and

$$\vec{y}^{\perp} := \vec{y} - \text{proj}_L(\vec{y})$$

the component of $\vec{y}$ **orthogonal to** L (or $\vec{w}$).

For these two vectors one obtains

$$\vec{y} = \text{proj}_L(\vec{y}) + \vec{y}^{\perp} \qquad \text{and} \qquad \vec{w}\cdot\vec{y}^{\perp} = 0.$$

Verification: The first one is obviously from definition of $\vec{y}^{\perp}$.

$$\vec{w}\cdot\vec{y}^{\perp} = \vec{w}(\vec{y}-\text{proj}_L(\vec{y})) = \vec{w}\cdot\vec{y}-\vec{w}\cdot\text{proj}_L(\vec{y})) = \vec{w}\cdot\vec{y}-\vec{w}\cdot\left(\frac{\vec{y}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$
$$= \vec{w}\cdot\vec{y}-\left(\frac{\vec{y}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)(\vec{w}\cdot\vec{w}) = \vec{w}\cdot\vec{y}-\vec{y}\cdot\vec{w} = 0$$

Example 5. Let L be the line in $\mathbb{R}^3$ that is the span of vector $\vec{u} = \begin{bmatrix}1\\1\\2\end{bmatrix}$. Find a decomposition of the vector $\vec{y} = \begin{bmatrix}1\\2\\6\end{bmatrix}$ as $\vec{y} = \text{proj}_L(\vec{y}) + \vec{y}^{\perp}$. Here $\text{proj}_L(\vec{y})$ is the orthogonal projection of $\vec{y}$ onto L and $\vec{y}^{\perp}$ is the component of $\vec{y}$ orthogonal to L.

$$\text{proj}_L(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\cdot\vec{u} = \frac{1+2+12}{1+1+4}\begin{bmatrix}1\\1\\2\end{bmatrix} = \frac{15}{6}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}5/2\\5/2\\5\end{bmatrix}$$

$$\vec{y}^{\perp} = \vec{y} - \text{proj}_L(\vec{y}) = \begin{bmatrix}1\\2\\6\end{bmatrix} - \begin{bmatrix}5/2\\5/2\\5\end{bmatrix} = \begin{bmatrix}-3/2\\-1/2\\1\end{bmatrix}$$

$$\vec{y} = \begin{bmatrix}5/2\\5/2\\5\end{bmatrix} + \begin{bmatrix}-3/2\\-1/2\\1\end{bmatrix}$$

Example 6. Let L be the line in $\mathbb{R}^2$ that is the span of vector $\vec{w} = \begin{bmatrix}1\\2\end{bmatrix}$. Find a decomposition of the vector $\vec{y} = \begin{bmatrix}4\\3\end{bmatrix}$ as $\vec{y} = \text{proj}_L(\vec{y}) + \vec{y}^{\perp}$.

$$\text{proj}_L(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \left(\frac{4+6}{1+4}\right)\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$y^{\perp} = \vec{y} - \text{proj}_L(\vec{y}) = \begin{bmatrix}4\\3\end{bmatrix} - \begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\vec{y} = \begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix}$$

Suppose $\vec{w} \in \mathbb{R}^n$ is a vector on line L and $\vec{u} = \frac{\vec{w}}{||\vec{w}||}$ is the **unit** vector on L.

Theorem.

The orthogonal projection $T(\vec{x}) = \text{proj}_L(\vec{x})$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Proof: We show that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$.
We know that $\text{proj}_L(\vec{x}) := \left(\dfrac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$.
$T(\vec{u}+\vec{v}) = \text{proj}_L(\vec{u}+\vec{v}) = \left(\dfrac{(\vec{u}+\vec{v})\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \left(\dfrac{\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \left(\dfrac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} + \left(\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \text{proj}_L(\vec{u}) + \text{proj}_L((\vec{v}) = T(\vec{u}) + T(\vec{v})$.
Similarly, we can verify $T(c\vec{u}) = cT(\vec{u})$. So, T is linear.

Hence, the matrix of the orthogonal projection is $[\text{proj}_L(\vec{e}_1)\ \ \text{proj}_L(\vec{e}_2)\ \ \cdots\ \ \text{proj}_L(\vec{e}_n)]$.

In the case when $n = 2$, suppose $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$ and the unit vector is $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$

Theorem.

The **matrix** of the orthogonal projection $\text{proj}_L(\vec{x})$ is given by

$$P = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$$

Proof: We know that can can use the standard vectors $\vec{e}_1$ and $\vec{e}_2$ to find the matrix $P = [\text{proj}_L(\vec{e}_1)\quad \text{proj}_L(\vec{e}_2)]$.

$$\text{proj}_L(\vec{e}_1) = \left(\frac{\vec{e}_1\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \left(\frac{1w_1 + 0w_2}{w_1^2 + w_2^2}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 \\ w_1w_2 \end{bmatrix}$$

Similarly, we can calculate the second column $\text{proj}_L(\vec{e}_2)$.

Example 7. Find the matrix A for the orthogonal projection onto $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$P = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 8. Find the matrix A for the orthogonal projection onto the line L defined by the span of $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$P = \frac{1}{2^2+3^2}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{bmatrix}$$

Orthogonal projection onto a plane*.

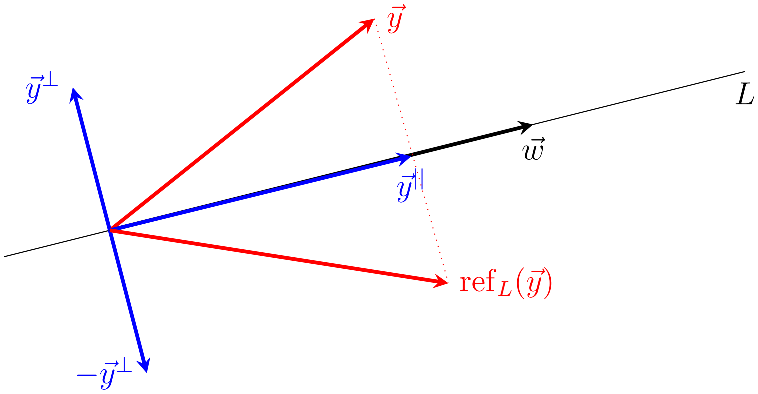
Example 9. *Let V be the plane defined by $x_1 + x_2 + 2x_3 = 0$ and let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$. Find a decomposition of $\vec{x} = \vec{y} + \vec{z}$ such that $\vec{y}$ on plane V and $\vec{z}$ is perpendicular to V.

The vector perpendicular to plane $x_1 + x_2 + 2x_3 = 0$ is $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

From Example 5, $\mathrm{proj}_{\vec{w}}\vec{x} = \begin{bmatrix} 5/2 \\ 5/2 \\ 5 \end{bmatrix} = \vec{z}$

$$\vec{y} = \vec{x} - \vec{z} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

3. Reflection



Definition. (Reflection along a line)

Let $\vec{w}$ be a nonzero vector in $\mathbb{R}^2$ and let $L = \text{Span}\{\vec{w}\}$ be the line in $\mathbb{R}^2$ spanned by $\vec{w}$. For a given vector $\vec{y} \in \mathbb{R}^2$, the vector

$$\text{ref}_L(\vec{y}) := \vec{y}^{||} - \vec{y}^{\perp}$$

is called the **reflection of $\vec{y}$ about** L. An equivalent formula is given by

$$\text{ref}_L(\vec{y}) := 2\,\text{proj}_L(\vec{y}) - \vec{y}$$

Theorem.

The reflection $T(\vec{x}) = \text{ref}_L(\vec{x})$ is a linear transformation. The matrix of T is given by

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a^2 + b^2 = 1$

Here, $\begin{bmatrix} a \\ b \end{bmatrix}$ is given by $\text{ref}_L(\vec{e}_1)$, where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\begin{bmatrix} b \\ -a \end{bmatrix}$ is given by $\text{ref}_L(\vec{e}_2)$, where $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 10. Find the matrix A of the reflection about $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{ref}_L(\vec{e}_1) = 2\text{proj}_L(\vec{e}_1) - \vec{e}_1 = 2\left(\frac{1}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So the matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Example 11. Find the matrix of the reflection about the line L defined by the span of $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$\text{proj}_L(\vec{e_1}) = \left(\frac{\vec{u}\cdot\vec{e_1}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \frac{2}{4+9}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix}$$

$$\text{ref}_L(\vec{e_1}) = 2\,\text{proj}_L(\vec{e_1}) - \vec{e_1} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{13} \\ 12/13 \end{bmatrix}$$

the matrix is $\begin{bmatrix} -\frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix}$

Example 12. (Example 5 continued) Let L be the line in $\mathbb{R}^3$ that is the span of vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ about the line L.

$$\text{ref}_L(\vec{y}) = 2\,\text{proj}_L(\vec{y}) - \vec{y}$$

$$= 2\begin{bmatrix} 5/2 \\ 5/2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

4. Rotation

Theorem. (Rotation)

The matrix of a counter-clockwise **rotation** in $\mathbb{R}^2$ trough an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The matrix is of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a^2 + b^2 = 1$.

Example 13. Find the matrix of a counter-clockwise **rotation** in $\mathbb{R}^2$ trough an angle $\theta = \pi/3$.

$$\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

5. Rotation combined with a scaling

Theorem. (Rotation combined with a scaling)

The matrix is of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

defines a linear transformation representing a counter-clockwise rotation of degree θ with a scaling k.
Here, $k = \sqrt{a^2 + b^2}$ and $\cos\theta = \frac{a}{k}$ and $\sin\theta = \frac{b}{k}$.

Example 14. Describe the geometric meaning of the transformation defined by the matrix

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$k = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{k} = \frac{\sqrt{3}}{2} \\ \sin\theta &= \frac{b}{k} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \theta = \frac{\pi}{6}$$

The matrix defines a linear transformation representing a counter-clockwise rotation of degree $\theta = \frac{\pi}{6}$ with a scaling $k = 2$.

Example 15. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which begins with first **reflection** ϕ about the line $y = x$, followed by **orthogonal projection** onto the line $y = -x$. Find the matrix of T.

Step 1. A vector on Line $y=x$ is $\vec{u}=\begin{bmatrix}1\\1\end{bmatrix}$.

$\text{ref}_{\vec{u}}(\vec{e}_1)=2\,\text{proj}_{\vec{u}}(\vec{e}_1)-\vec{e}_1=2\left(\frac{1}{2}\right)\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$

The matrix A of the reflection ϕ is

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Step 2. A vector on Lin $y=-x$ is $\vec{v}=\begin{bmatrix}1\\-1\end{bmatrix}$

The matrix B of orthogonal projection is $B=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

Step 3 The matrix of T is

$$BA=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}$$

Example 16. $T:\mathbb{R}^2\to\mathbb{R}^2$ is the linear transformation which begins with first **rotation** ρ, counter-clockwise about the origin by angle $\theta=\frac{\pi}{4}$, followed by **orthogonal projection** onto the line $y=2x$. Find the matrix of T.

Step 1:

The matrix A of rotation ρ is

$$\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

Step 2:

A vector on $y=2x$ is $\vec{w}=\begin{bmatrix}1\\2\end{bmatrix}$.

The matrix B of orthogonal projection is $B=\frac{1}{1^2+2^2}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

Step 3 The matrix of T is given by

$$BA=\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}\frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{6\sqrt{2}}{2} & \sqrt{2}\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt{2}}{10} & \frac{\sqrt{2}}{10}\\ \frac{6\sqrt{2}}{10} & \frac{\sqrt{2}}{5}\end{bmatrix}$$

6. Shear*

Theorem.

The matrix of a **horizontal shear** is of the form

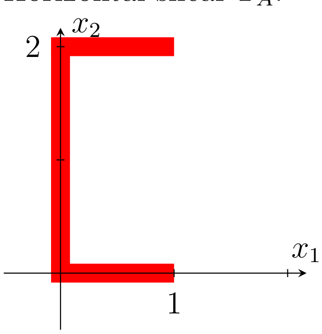
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

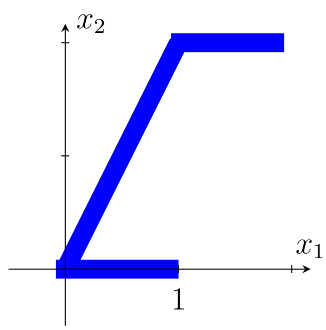
The matrix of a **vertical shear** is of the form

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Example 17. * Draw the graph of transformation defined by the following matrices: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Horizontal shear T_A:





Vertical shear T_B: