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Recommended learning order for chapter 2 is §2.3 → §2.1 → §2.4 → §2.2.

§2.1 Linear transformation and its inverse

Definition. Transformation

A **transformation** (or function or map) T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a rule of assigning to each vector $\vec{x} \in \mathbb{R}^m$ a new vector $T(\vec{x}) \in \mathbb{R}^n$.

To indicate that T is a transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ we write

$$T\colon \mathbb{R}^m \to \mathbb{R}^n.$$

We call $\mathbb{R}^m$ the **domain** of T. We call $\mathbb{R}^n$ the **codomain** of T. The set of all vectors $T(\vec{x})$ in $\mathbb{R}^n$, for all $\vec{x} \in \mathbb{R}^m$, is called the ***range*** or ***image*** of T, denoted as $\mathrm{im}(T) = \{T(\vec{x}) | \vec{x} \in \mathbb{R}^m\}$

Definition. (Linear transformation)

A transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is called **linear** if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A \cdot \vec{x}$$

Example 1. What are the transformations defined by the following matrices?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

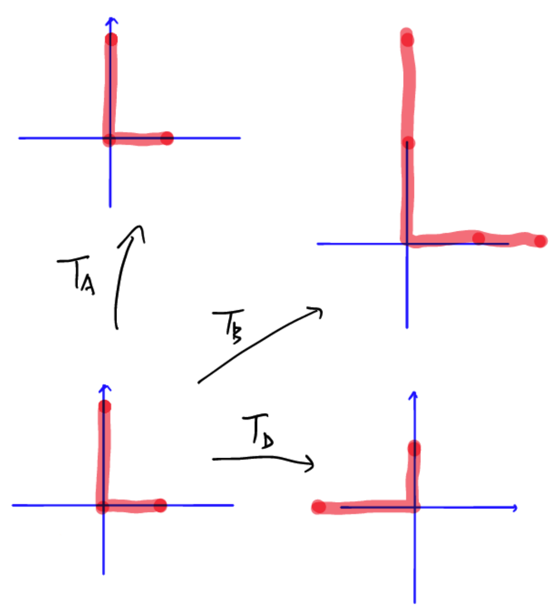
$$A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$B\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$D\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$



Example 2. What is the transformation T defined from matrix $A = \begin{bmatrix} 1 & 4 & -3 & -1 \\ 2 & 9 & -3 & -8 \\ 2 & 11 & 6 & -26 \end{bmatrix}$?

$$A\vec{x} = \begin{bmatrix} 1 & 4 & -3 & -1 \\ 2 & 9 & -3 & -8 \\ 2 & 11 & 6 & -26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 - 3x_3 - x_4 \\ 2x_1 + 9x_2 - 3x_3 - 8x_4 \\ 2x_1 + 11x_2 + 6x_3 - 26x_4 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 11 \end{bmatrix} \qquad A\vec{e}_3 = \begin{bmatrix} -3 \\ -3 \\ 6 \end{bmatrix} \qquad A\vec{e}_4 = \begin{bmatrix} -1 \\ -8 \\ -26 \end{bmatrix}$$

Denote $\vec{e}_1$, $\vec{e}_2$, ... , $\vec{e}_m$ be the column vectors of the identity matrix I_m. We call them the **standard** vectors in $\mathbb{R}^m$.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$ can be written as $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m$

Theorem.

A transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if and only if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^m$.

(ii) $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$ for all $\vec{u} \in \mathbb{R}^m$ and all $c \in \mathbb{R}$.

Using this theorem, we can directly verify a transformation to be linear without knowing its matrix. We will use it in §2.2 for projection transformations and reflection transformations.

Proof: " $\Rightarrow$" If T is linear, then by definition, there is an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$. For any vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^m$ and c is a scalar, then
(a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
(b)$A(c\vec{u}) = c(A\vec{u})$.

" $\Leftarrow$" Suppose there is a transformation satisfies the two conditions (i) and (ii). We need to show that it is linear. For any $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m \in \mathbb{R}^m$,

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m) \\ &= [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_m)]\vec{x} \\ &= A\vec{x} \end{aligned}$$

A by-product of the proof is that we can explicitly find the matrix A of a linear transformation. We write it as the following theorem.

Theorem.

Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then the matrix of T is given by

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_m)].$$

Proposition.

If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then
(i) $T(\vec{0}) = \vec{0}$;
(ii) $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_pT(\vec{v}_p)$ for all scalars $c_1, \ldots, c_p \in \mathbb{R}$ and all vectors $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^m$.

Example 3. Is $\begin{cases} y_1 = x_1 + 1 \\ y_2 = 2x_1 + x_2 \end{cases}$ a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?

$T(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So it is not a linear transformation.

Example 4. Is $\begin{cases} y_1 = x_1^2 \\ y_2 = x_1 + x_2 \end{cases}$ a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?

$T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\ne$
$2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. So it is not a linear transformation.

Example 5. Let T be a linear transformation from $\mathbb{R}^n \to \mathbb{R}^2$. If $\vec{w} = 2\vec{u} - 3\vec{v} \in R^n$, $T(\vec{u}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find $T(\vec{w})$.

$$T(\vec{w}) = T(2\vec{u} - 3\vec{v}) = 2\,T(\vec{u}) - 3\,T(\vec{v}) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Definition.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **invertible** if for any $\vec{y} \in \mathbb{R}^n$ there is a unique $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

If A is the matrix for T, then T is invertible if and only if $\textbf{rref}(A) = I_n$.

Example 6. Find the inverse of the transformation defined by matrix $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. After we learned §2.4, we can do this example very easily.

$$\left[\begin{array}{cc|c} 1 & 3 & y_1 \\ 2 & 5 & y_2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & y_1 \\ 0 & -1 & -2y_1 + y_2 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 3 & y_1 \\ 0 & 1 & 2y_1 - y_2 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & -5y_1 + 3y_2 \\ 0 & 1 & 2y_1 - y_2 \end{array}\right]$$

So $\begin{cases} x_1 = -5y_1 + 3y_2 \\ x_2 = 2y_1 - y_2 \end{cases}$

unique solution for any $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Suppose we already learned §2.3.

Theorem. [Matrix product and composition of transformations]

Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then the product AB is the matrix of the transformation composition $T_A \circ T_B$.

Proof. $(T_A \circ T_B)(\vec{x}) = T_A(B\vec{x}) = A(B\vec{x}) = AB\vec{x}$.