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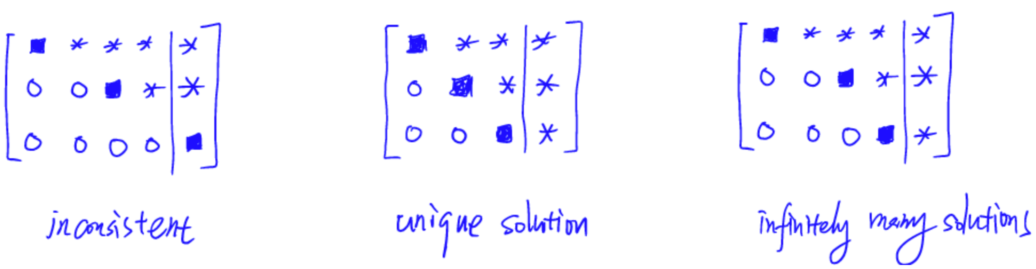
§1.3 On The Solution of Linear Systems; Matrix Algebra

A linear system is said to be **consistent** if there is at least one solution. It is **inconsistent** if there is no solution.

> **Theorem.** Number of solutions of a linear system
>
> A linear system is inconsistent *if and only if* a row-echelon from(**ref**) of the augmented matrix has a row
>
> $$[\ 0\ 0\ 0\ \ldots\ 0 \mid b\ ] \text{ with } b \neq 0.$$
>
> Moreover, if a linear system is consistent, it has either
> • a unique solution (no free variables), or
> • infinitely many solutions (at least one free variable).

Examples:

Denote the square box as non-zero number.



Example 1. Find all possible h and k such that the linear system

$$\begin{cases} x_1 + hx_2 = 1 \\ 3x_1 + 12x_2 = k \end{cases}$$

has **no** solution.

> $$\begin{bmatrix} 1 & h & 1 \\ 3 & 12 & k \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & h & 1 \\ 3 & 12 - 3h & k - 3 \end{bmatrix}$$
>
> So, by the above theorem, the linear system has no solution only if $12 - 3h = 0$ and $k - 3 \neq 0$, that is, $k = 4$ and $k \neq 3$.

> **Definition.**
>
> The **rank of a matrix** A is the number of the leading entries in either **ref**(A) or **rref**(A), denoted by rank(A).

Consider a linear system of n equations with m variables. That is the coefficient matrix A is of size $n \times m$.

Use the rank A="number of pivots"=m-"number of free variables", we can get the following properties.

Proposition.

Consider a linear system with coefficient matrix A is of size $n \times m$.

(1.) $\text{rank}(A) \leq n$ and $\text{rank}(A) \leq m$.

(2.) If the system is inconsistent, then $\text{rank}(A) < n$.

(3.) If the system has exactly one solution, then $\text{rank}(A) = m$.

(4.) If the system has infinitely many solutions, then $\text{rank}(A) < m$.

Remark: Try to understand the following properties. Do not try to memorize them literally. Use the **ref** or **rref** of A to understand each property.
(1). The number of pivots is smaller or equal the number of rows or columns.
(2). If the system is inconsistent, then there is a row of form $[0\ 0\ \cdots\ 0\ 1]$ in the **rref** of the augmented matrix. So, there is a row of zeros in **rref**(A), so rank(A) is smaller than the number of rows.
(3). If the system has exactly one solution, then each variable must be on a pivot column, so, the number of variables (m) = the number of pivots (rank(A)).
(4). If the system has infinitely many solutions, then there is at least one free variable. So there are more variables than pivots.

Consequently, (using (3.) and (1.))
(a). If the system has exactly one solution, then the number of variables must be not more than the number of equations ($m \leq n$).
(b). If a linear system has more variables than equations ($m > n$), then it has either no solution or infinitely many solutions.

Items (a) and (b) are contrapositive.

Review a little of logic statements:

Consider any conditional statement (s): If H, then C.
The **contrapositive** of (s): If not C, then not H.

Contrapositive of any true statement is also true.
Contrapositive of any false statement is also false.

The **converse** of (s) is: If C, then H.
The **inverse** of (s) is: If not H, then not C.

The converse or inverse of a true conditional statement (s) **may** be false.

The inverse of (s) and the converse of (s) are contrapositive.

Example 2. Consider the converse and inverse of (a) and determine whether or not they are true or false.

The converse of (a): If the number of variables is not more than the number of equations, then the system has exactly one solution.
The inverse of (a): If the system has no solution or infinitely many solutions, then the number of variables must be more than the number of equations ($m > n$).
Both are false, for example, the augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$ has infinitely many solutions.

Proposition.

A linear system with an $n \times n$ coefficient matrix A has exactly one solution if and only if $\text{rank}(A) = n$. In this case $\textbf{rref}(A) = I_n$.

▶ Matrix Algebra

Definition.

- The ***sum*** $A + B$ of $n \times m$ matrices A and B is the new $n \times m$ matrix obtained by adding corresponding entries of A and B.
- The ***scalar product*** $r \cdot A$ of a scalar (real number) r and a matrix A is the matrix obtained by multiplying each entry of A by r.

Example 3. Given the matrices A, B, C,

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -1 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -2 & 6 \end{bmatrix}$$

Compute $A + B$, $3A$ and $A + C$.

$$A + B = \begin{bmatrix} 1+2 & -2+0 \\ -3-1 & 7-5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3(1) & 3(-2) \\ 3(-3) & 3(7) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -9 & 21 \end{bmatrix}$$

$$A + C = \text{Undefined.}$$

Theorem. Algebraic properties of matrix operations

For $n \times m$ matrices A, B, C and scalar r, s, the following hold.

(1) $A + B = B + A$;

(2) $(A + B) + C = A + (B + C)$;

(3) $A + 0 = A$;

(4) $A + (-A) = 0$;

(5) $r(A + B) = rA + rB$;

(6) $(r + s)A = rA + sA$;

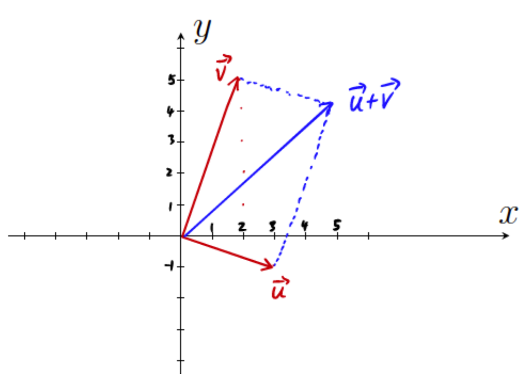
(7) $r(sA) = (rs)A$;

(8) $1A = A$.

Because vectors are special matrices, the operations sum and scalar products are also defined.
In particular, we denote $-\vec{v}$ for $(-1) \cdot \vec{v}$.
Moreover, they have geometric meanings.

Example. Sum. (Parallelogram Rule)

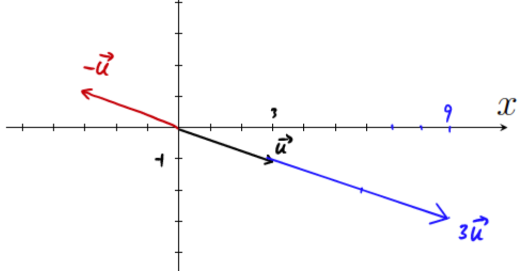
$$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$



Example. Scalar products $3\vec{u}$ and $-\vec{u}$.

$$3\vec{u} = 3 \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}.$$

$$-\vec{u} = -1 \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$



Recall from §1.2 H.W. 36,

Definition.

The **dot product** of two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is defined as

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$$

Remarks: 1. $\vec{u}$ and $\vec{v}$ must of the same size. 2. The dot product is not row-column-sensitive.

However, for generalization, we prefer to denote it as $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Example 4. $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$.

The dot product $\vec{u} \cdot \vec{v} = 1(-1) + 2(3) + 3(2) = 11$.

- **Product of a matrix A and a vector $\vec{x}$.**

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$ be an $n \times m$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ and rows $R_1, R_2, \ldots, R_n$. Let $\vec{x}$ be a vector in $\mathbb{R}^m$.

Definition.

The ***product*** of A and $\vec{x}$ defined to be

$$A\vec{x} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \vec{x} = \begin{bmatrix} R_1 \cdot \vec{x} \\ R_2 \cdot \vec{x} \\ \vdots \\ R_n \cdot \vec{x} \end{bmatrix}$$

Theorem.

The ***product*** of A and $\vec{x}$ can also be computed as

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m.$$

Verification of the theorem: For $n \times m$ matrix A, each formula gives us the result $A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix}$

Remark: The above two formulas are equivalent. The 1st formula is better for computation. The 2ed formula is better for theory.

Example 5.

From Definition 1.

$$A\vec{x} = \begin{bmatrix} -5 & -5 & -3 \\ 0 & -4 & -4 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} (-5)(-1) + (-5)(2) + (-3)(-3) \\ (-4)(2) + (-4)(-3) \\ (1)(-1) + (1)(2) + (4)(-3) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -11 \end{bmatrix}$$

From Definition 2.

$$A\vec{x} = \begin{bmatrix} -5 & -5 & -3 \\ 0 & -4 & -4 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = (-1)\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} -3 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -11 \end{bmatrix}$$

Or more generally,

$$A\vec{x} = \begin{bmatrix} -5 & -5 & -3 \\ 0 & -4 & -4 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-5)(x_1) + (-5)(x_2) + (-3)(x_3) \\ (-4)(x_2) + (-4)(x_3) \\ (1)(x_1) + (1)(x_2) + (4)(x_3) \end{bmatrix}$$

- **Linear combination**

Definition.

A vector $\vec{b}$ in $\mathbb{R}^n$ is called ***linear combination*** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, x_2, ..., x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$$

Example 6. Is $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ 15 \end{bmatrix}$ a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$?

Is $x_1\vec{u} + x_2\vec{v} = \vec{b}$ has a solution?

Is $\begin{bmatrix} x_1 \\ -2x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 15 \end{bmatrix}$ has a solution?

Is $\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ 5x_1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 15 \end{bmatrix}$ has a solution?

Is $\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ 5x_1 = 15 \end{cases}$ has a solution?

$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ 5 & 0 & 15 \end{array}\right]$ **rref** $= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. So, yes.

Theorem.

Let A be an $n \times m$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$, and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Then the matrix equation

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b},$$

which has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n & \vec{b} \end{bmatrix}.$$

Example.

Matrix equation

$$A\vec{x} = \begin{bmatrix} -5 & -5 & -3 \\ 0 & -4 & -4 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_1 - 5x_2 - 3x_3 \\ -4x_2 - 4x_3 \\ x_1 + x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -11 \end{bmatrix}$$

Vector equation:

$$x_1 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -11 \end{bmatrix}$$

Linear system and Augmented matrix:

$$\begin{cases} -5x_1 - 5x_2 - 3x_3 = 4 \\ -4x_2 - 4x_3 = 4 \\ x_1 + x_2 + 4x_3 = -11 \end{cases} \qquad \begin{bmatrix} -5 & -5 & -3 & 4 \\ 0 & -4 & -4 & 4 \\ 1 & 1 & 4 & -11 \end{bmatrix}$$

Theorem. (Algebraic Rules for $A\vec{x}$)

If A is an $n \times m$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^m$ and c is a scalar, then
(a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
(b)$A(c\vec{u}) = c(A\vec{u})$.

Proof: These two formulas can be verified by direct calculation. (To make it easy, suppose $m = n = 3$.)

$$A(\vec{u}+\vec{v}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(u_1+v_1) + a_{12}(u_2+v_2) + a_{13}(u_3+v_3) \\ a_{21}(u_1+v_1) + a_{22}(u_2+v_2) + a_{23}(u_3+v_3) \\ a_{31}(u_1+v_1) + a_{32}(u_2+v_2) + a_{33}(u_3+v_3) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(u_1) + a_{12}(u_2) + a_{13}(u_3) \\ a_{21}(u_1) + a_{22}(u_2) + a_{23}(u_3) \\ a_{31}(u_1) + a_{32}(u_2) + a_{33}(u_3) \end{bmatrix} + \begin{bmatrix} a_{11}(v_1) + a_{12}(v_2) + a_{13}(v_3) \\ a_{21}(v_1) + a_{22}(v_2) + a_{23}(v_3) \\ a_{31}(v_1) + a_{32}(v_2) + a_{33}(v_3) \end{bmatrix}$$

$$= A\vec{u} + A\vec{v}.$$

Question.

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ 5 & 0 \end{bmatrix}$$

Is the matrix equation $A\vec{x} = \vec{b}$ have a solution for every $\vec{b} \in \mathbb{R}^3$?

$\textbf{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

So, it is possible that $\textbf{rref}(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the last row is a contradiction. For example, when $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ 10 \end{bmatrix}$.