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About Linear Algebra

Solving a linear system, computing the sum, product, inverse of matrices, calculating eigenvalues and eigenvectors of a matrix are some basic topics in linear algebra. However, linear algebra is much more than that.

Linear algebra has three sides: concepts, computational techniques, and applications. This course (Math 2331) is the first level of linear algebra. One of the goals of this course is to help you master the subject and see the interplay among them. The material presented in this course involves theorems (few proofs), formulas, and computations of various.

Main objects:

1. System of linear equations.

2. Vectors and matrices.

3. Linear spaces.

We will learn

1. Concepts and notations of the objects and operations;

2. Properties of the operations;

3. Algorithms of operations;

Applications:

1. Engineering, e.g., traffic volumes, equilateral truss, a line of springs.

2. Large data set and Statistics, e.g., least squares data fitting, linear regression.

3. Probability, e.g., Markov chains (Dynamical systems).

4. Machine learning. e.g., translation, rescaling and rotation of images. Digital image compression using singular value decomposition(SVD).

5. Economics, supply chain.

6. Computer computations. Linear Programming, the simplex optimization method.

7. differential equations

8. fast Fourier transform, networks, game theory...

Linear algebra is the most widely-used mathematics tool in engineering, applied science, and statistics. Unlike the one-variable calculus problems that you can solve by hand calculation (or with the aid of a graphing calculator), linear algebra algorithms generally require substantial computer resources.

Popular computer programming languages:

1. Matlab; 2. Python; 3. Maple; 4. Mathematica; 5. ...

(Matlab: **matrix** laboratory.)
(Google's Python package: **Tensor**Flow for machine learning.)

Two future courses of linear algebra are: Math 4571 Advanced Linear Algebra is an undergraduate linear algebra with proofs and more general topics and Math 5110 Applied Linear Algebra is graduate course with proofs and more real world applications using Matlab. The best software package for this purpose is generally agreed to be MATLAB.

The theory of linear algebra is very well understood by mathematicians, however, there are still many interesting areas of research involving linear algebra and questions of computation.

If we pass to systems of equations that are of degree two or higher, then the mathematics is much more difficult and complex. This area of study is known as algebraic geometry, which is one of the most popular and hard research field in pure mathematics now.

§1.1 Introduction to Linear Systems

Review some backgrounds

• **Numbers:**

Natural numbers $\mathbb{N}$. (Closed under addition and multiplication)

Integers $\mathbb{Z}$. (Closed under addition and multiplication, and subtraction.)

Rational numbers $\mathbb{Q}$. (Closed under addition, multiplication, and subtraction, and division.)

Real numbers $\mathbb{R}$. (One-to-one corresponding with points on a line, e.g., irrational numbers $\sqrt{2}$, π, e, etc.)

Complex numbers $\mathbb{C}$. (One-to-one corresponding with points on a plane. Fundamental theorem of algebra: Any polynomial equation of degree n has n complex solutions.)

• **Single variable functions:**(Calculus1,2)

Linear function: $f(x) = ax + b$

Non-linear function examples: $f(x) = x^2 + 2x$; $f(x) = e^x + \sin x$

• **Two variables functions:**(Calculus 3)

Linear function: $f(x_1, x_2) = a_1x_1 + a_2x_2 + c$

Non-linear function examples: $f(x) = x_1^2 + e^{x_2} + x_1 + 2$

• **Equations:** $f(x_1, x_2) = g(x_1, x_2)$.

Linear equation examples: $2x_1 + 3x_2 = 0$; $2x_1 - 3x_2 = 1$;

Some basic notations:

Notation	Meaning
$\mathbb{N}$	$\{0, 1, 2, \dots\}$ The *natural numbers.*
$\mathbb{Z}$	$\{\dots, -1, 0, 1, 2, \dots\}$ The *integers.*
$\mathbb{Q}$	$\{p/q \mid p, q \in \mathbb{Z}$ and $q \neq 0\}$ The *rationals.*
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
$x \in S$	"x is a member of S"
$\{x \in S \mid P(x)\}$	"the set of all x in S such that $P(x)$"
$P \implies Q$	"P implies Q"
$P \iff Q$	"P if and only if Q"
$\forall x$	"for all x"
$\exists x$	"there exists an x"
s.t.	"such that"

Conditional Statement	If P, then Q.	$P \Rightarrow Q$
Converse	If Q, then P	$Q \Rightarrow P$
Inverse	If not P, then not Q	$P \Rightarrow\ Q$
Contrapositive	If not Q, then not P	$Q \Rightarrow\ P$
Biconditional	P if and only if Q	$P \Leftrightarrow Q$

Linear Equations

Definition. (linear equation)

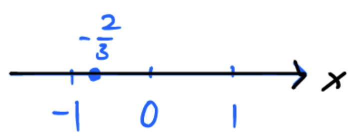
A ***linear equation*** in the variables $x_1, x_2, x_3, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

Here, $a_1, a_2, \ldots, a_n$, and b are real numbers.
The real numbers $a_1, a_2, \ldots, a_n$ are called the ***coefficients*** of the linear equation.

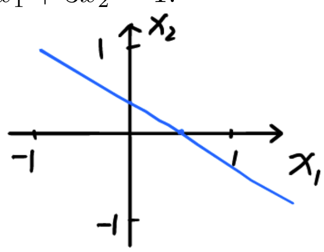
Example 1. $3x + 2 = 0$;

$x = -\dfrac{2}{3}$.



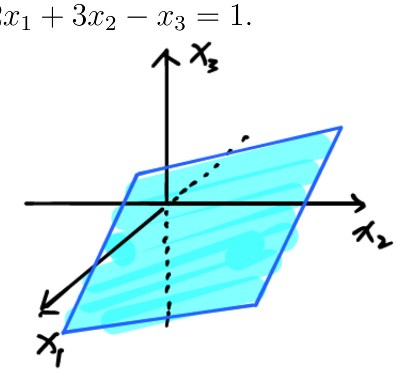
Example 2. $2x_1 + 3x_2 = 1$;

Solutions include $(0, \frac{1}{3})$, $(0.5, 0)$, $(-1, 1)$, ...
More precisely, the set of all solutions of the equation is the line in $\mathbb{R}^2$ defined by $2x_1 + 3x_2 = 1$.



Example 3. $2x_1 + 3x_2 - x_3 = 1$;

Solutions include $(0, 0, -1)$, $(1, 0, 1)$, $(0, 1, 2)$, ...
More precisely, the set of all solutions of the equation is the plane in $\mathbb{R}^3$ defined by $2x_1 + 3x_2 - x_3 = 1$.



Definition. (linear system)

A **system of linear equations** (or ***linear system***) is a collection of linear equations in the same variables $x_1, x_2, \ldots, x_n$.

Example 4. $\begin{cases} -2x_1 + x_2 = 2 \\ x_1 - 2x_2 = 2 \end{cases}$

$$\begin{matrix} L_1 \\ L_2 \end{matrix}\begin{cases} -2x_1 + x_2 = 2 \\ x_1 - 2x_2 = 2 \end{cases}$$

$$\begin{matrix} L_1 \\ 2L_2 \end{matrix}\begin{cases} -2x_1 + x_2 = 2 \\ 2x_1 - 4x_2 = 4 \end{cases}$$

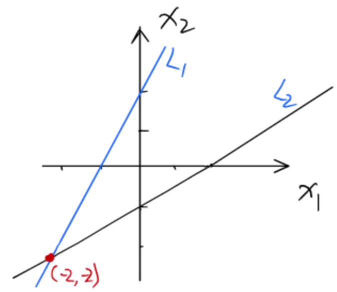
$x_2 = -2$

$x_1 = -2$

$$L_2 + L_1 \begin{cases} -2x_1 + x_2 = 2 \\ -3x_2 = 6 \end{cases}$$

eliminate x_1

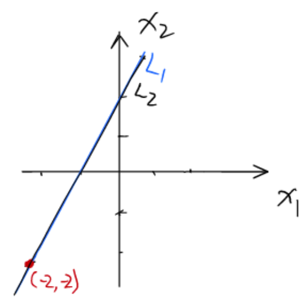
$(-2, -2)$ is a solution.



Example 5. $\begin{cases} -2x_1 + x_2 = 2 \\ 4x_1 - 2x_2 = -4 \end{cases}$

$$L_2 + 2L_1 \begin{cases} -2x_2 + x_2 = 2 \\ 0 = 0 \end{cases}$$

All points in L_1 are solutions for the linear system

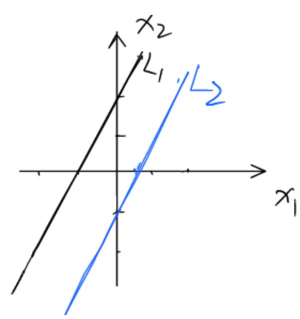


Example 6. $\begin{cases} -2x_1 + x_2 = 2 \\ -2x_1 + x_2 = -1 \end{cases}$

$$L_2 - L_1 \begin{cases} -2x_1 + x_2 = 2 \\ 0 = -3 \end{cases}$$

contradiction!

No solution.



A **linear** system has either one solution, or infinitely many solutions, or no solution.

Geometric interpretation for linear systems with 3 variables x_1, x_2, x_3.

(a) (b) (c)

Example 7. Solve $\begin{cases} x_1 - 3x_2 - 5x_3 = 1 \\ x_1 - x_2 - 2x_3 = 0 \\ 3x_1 - x_2 + x_3 = 3 \end{cases}$

$$\begin{array}{l} \\ L_2 - L_1 \\ L_3 - 3L_1 \end{array} \begin{cases} x_1 - 3x_2 - 5x_3 = 1 \\ 2x_2 + 3x_3 = -1 \\ 8x_2 + 16x_3 = 0 \end{cases}$$

$$\begin{array}{l} \\ \\ L_3 - 4L_2 \end{array} \begin{cases} x_1 - 3x_2 - 5x_3 = 1 \\ 2x_2 + 3x_3 = -1 \\ 4x_3 = 4 \end{cases}$$

$$\begin{array}{l} \\ \\ \frac{1}{4}L_3 \end{array} \begin{cases} x_1 - 3x_2 - 5x_3 = 1 \\ 2x_2 + 3x_3 = -1 \\ x_3 = 1 \end{cases}$$

$$\begin{array}{l} L_1 + 5L_3 \\ L_2 - 3L_3 \\ \\ \end{array} \begin{cases} x_1 - 3x_2 = 6 \\ 2x_2 = -4 \\ x_3 = 1 \end{cases}$$

$$\begin{array}{l} \\ \frac{1}{2}L_2 \\ \\ \end{array} \begin{cases} x_1 - 3x_2 = 6 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

$$\begin{array}{l} L_1 + 3L_2 \\ \\ \\ \end{array} \begin{cases} x_1 = 0 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

So, $(0, -2, 1)$ is the only solution for the linear system.

Example 8. Solve $\begin{cases} x_2 + 3x_3 = 7 \\ x_1 + 3x_2 + 2x_3 = -3 \\ 2x_1 + 5x_2 + x_3 = -1 \end{cases}$

$$L_1 \leftrightarrow L_2 \begin{cases} x_1 + 3x_2 + 2x_3 = -3 \\ x_2 + 3x_3 = 7 \\ 2x_1 + 5x_2 + x_3 = -1 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 + 2x_3 = -3 \\ x_2 + 3x_3 = 7 \\ -x_2 - 3x_3 = 5 & L_3 - 2L_1 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 + 2x_3 = -3 \\ x_2 + 3x_3 = 7 \\ 0 = 12 & L_3 + L_2 \end{cases}$$

contradiction

The system has no solution.

Example 9. Find all polynomials of the form $p(t) = a + bt + ct^2$ with graphs run through the point $(1, 3)$ such that $p'(1) = 5$ and $p''(1) = 4$.

$P'(t) = b + 2ct$

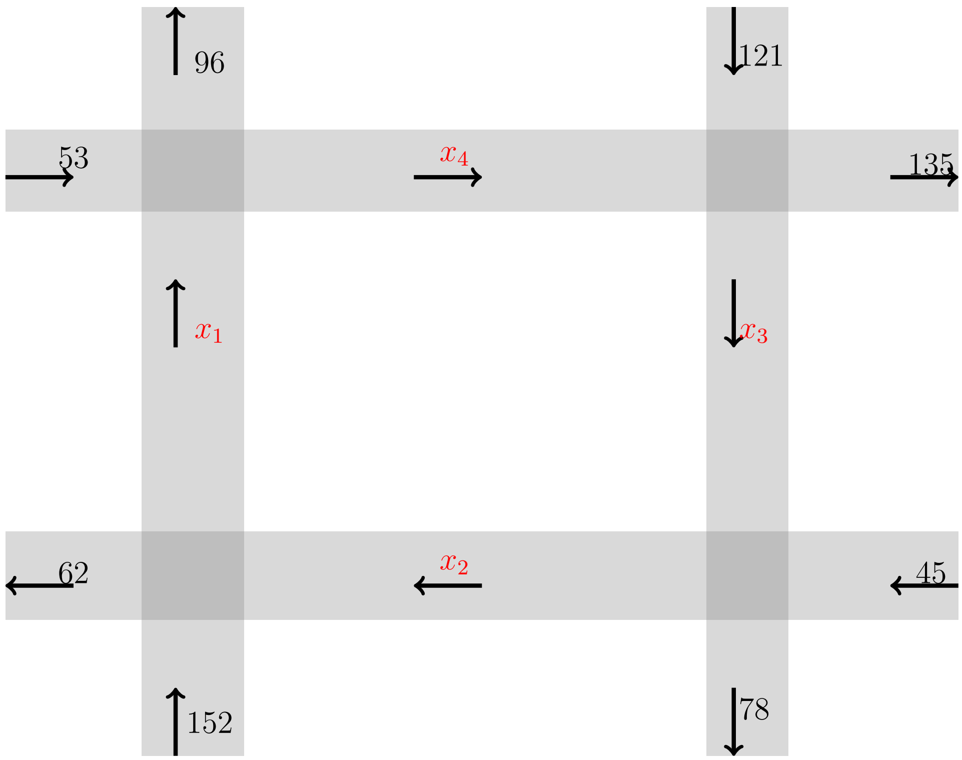
$P''(t) = 2c$

So $\begin{cases} a + b + c = 3 \\ b + 2c = 5 \\ 2c = 4 \end{cases} \Rightarrow \begin{matrix} a = 0 \\ b = 1 \\ c = 2 \end{matrix}$

Application Examples.

Focus on setting up the system of equations first in this section.

Example 10. (Traffic volumes) The following diagram shows part of the streets in Boston. We assume that the streets are one way, and that the average number of cars entering and leaving this section between 1 pm to 2 pm is given in the chart. Find the numbers of the traffic between each of four intersections.



For each intersection, In = Out.
We can set up the linear system

$$\begin{cases} x_1+53=x_4+96 \\ x_4+121=x_3+135 \\ x_3+45=x_2+78 \\ x_2+152=x_1+62 \end{cases} \quad \text{Rewrite as} \quad \begin{cases} x_1-x_4=43 \\ -x_3+x_4=14 \\ -x_2+x_4=33 \\ -x_1+x_2=-90 \end{cases}$$

We can solve the linear system using the techniques in the next section. (Wait until §1.2)

The augmented matrix of the linear system is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & 0 & -1 & 1 & 14 \\ 0 & -1 & 1 & 0 & 33 \\ -1 & 1 & 0 & 0 & -90 \end{bmatrix} \xrightarrow{R_4+R_1} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & 0 & -1 & 1 & 14 \\ 0 & -1 & 1 & 0 & 33 \\ 0 & 1 & 0 & -1 & -47 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & -1 & 1 & 0 & 33 \\ 0 & 0 & -1 & 1 & 14 \\ 0 & 1 & 0 & -1 & -47 \end{bmatrix}$$

$$\xrightarrow{R_4+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & -1 & 1 & 0 & 33 \\ 0 & 0 & -1 & 1 & 14 \\ 0 & 0 & 1 & -1 & -14 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & -1 & 1 & 0 & 33 \\ 0 & 0 & -1 & 1 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_2]{-R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & 1 & -1 & 0 & -33 \\ 0 & 0 & 1 & -1 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 43 \\ 0 & 1 & 0 & -1 & -47 \\ 0 & 0 & 1 & -1 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

$$\begin{cases} x_1 - x_4 = 43 \\ x_2 - x_4 = -47 \\ x_3 - x_4 = -14 \\ x_4 \text{ free} \end{cases} \qquad \begin{cases} x_1 = x_4 + 43 \\ x_2 = x_4 - 47 \\ x_3 = x_4 - 14 \\ x_4 \text{ free} \end{cases}$$

Analyzing:

Extra information: ① x_1, x_2, x_3, x_4 are positive integers.

② Since $x_2 = x_4 - 47 > 0$, we have $\boxed{x_4 > 47}$

③ $x_1 > x_4 > x_3 > x_2$

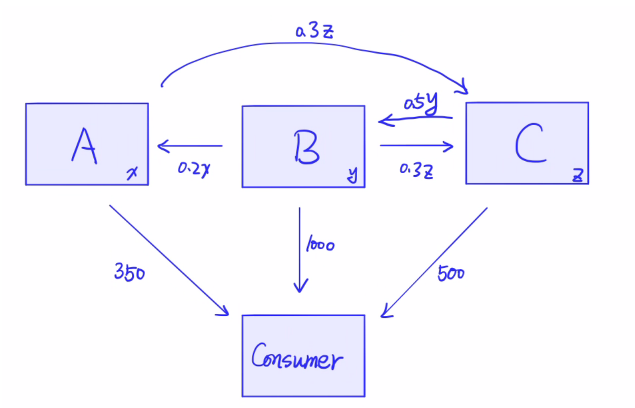
④ Total number in and out the four crossings are 371

⑤ $x_1 + x_2 + x_3 + x_4 = 4x_4 - 18 < 371 \times 2 \Rightarrow \boxed{x_4 < 190}$

Example: ① If $x_4 = 60$, then $x_1 = 103$, $x_2 = 13$, $x_3 = 46$

② If $x_4 = 100$, then $x_1 = 143$, $x_2 = 53$, $x_3 = 86$

Example 11. (Supply chain in Economy.) Consider a simple example of input-output analysis, an economy with only two industries, A, B and C. Assume that the consumer demand for their products is, respectively, 350, 1000 and 500 (in millions of dollars) per year. The interindustry demand needs to be considered as well. Suppose the interindustry demands are given as follows. Find the outputs x, y, z needed to satisfy both consumer and interindustry demands.



$$\begin{cases} x = 0.3z + 350 \\ y = 0.2x + 0.3z + 1000 \\ z = 0.5y + 500 \end{cases} \Rightarrow \begin{cases} x - 0.3z = 350 \\ -0.2x + y - 0.3z = 1000 \\ -0.5y + z = 500 \end{cases}$$

We can solve the linear system using the techniques in the next section.
The solution is $(x, y, z) = (29875/41, 62500/41, 51750/41)$

Example 12. (Integer solutions) I have 15 bills in my wallet, in the denominations of US\$ 1, 5, and 10, worth \$76 in total. How many do I have of each denomination?

Solution: Suppose x_1 =number of \$1 bills, x_2 =number of \$5 bills, and x_3 =number of \$10 bills. Hence, the linear system is

$$\begin{cases} x_1 + x_2 + x_3 = 15 \\ x_1 + 5x_2 + 10x_3 = 76 \end{cases}$$

We can solve the linear system using the techniques in the next section. (Here, we only want positive integer solutions.)

We can simply the linear system as $\begin{cases} x_1 = \frac{5}{4}x_3 - \frac{1}{4} \\ x_2 = -\frac{9}{4}x_3 + \frac{61}{4} \end{cases}$ where x_3 can be any numbers.

However, in our example, we know that x_3 can only be 1,2,3,4,5,6,7. Only when $x_3 = 5$, x_1 and x_2 have integer values $x_1 = 6$ and $x_2 = 4$.

Example 13. (Chemistry equation) Consider the following chemistry problem: x molecules of toluene, C_7H_8, together with y molecules of nitric oxide, HNO_3. Putting them together, one can produce z molecules trinitrotoluene (TNT), which has the form $C_7H_5O_6N_3$, with w molecules water (H_2O). One needs to balance the equation:

$$xC_7H_8 + yHNO_3 \to zC_7H_5O_6N_3 + wH_2O$$

Find values of x, y, z, and w so that the number of atoms of each type is the same before and after the reaction.

Solution: Counting them each atom, we can get the system of equations;

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

We also only care about positive integer solutions.
Reorganize the system

$$\begin{aligned} 7x - 7z &= 0 \\ 8x + 1y - 5z - 2w &= 0 \\ 1y - 3z &= 0 \\ 3y - 6z - 1w &= 0 \end{aligned}$$

From equation 1 and 3 we have $x = z$ and $y = 3z$, plug in back and solve it we have

Statistics Problem: Fitting a function of a certain type of data. We use the following example to illustrate this application.

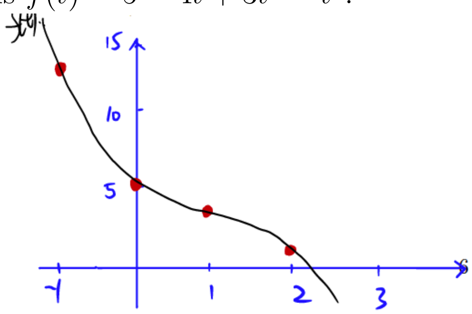
Example 14. Find a cubic polynomial $f(t) = c_0 + c_1t + c_2t^2 + c_3t^3$ whose graph passes through the points $(0, 5)$, $(1, 3)$, $(-1, 13)$, $(2, 1)$.

Solution:

We need to solve the linear system $\begin{cases} c_0 & = 5 \\ c_0 + c_1 + c_2 + c_3 & = 3 \\ c_0 - c_1 + c_2 - c_3 & = 13 \\ c_0 + 2c_1 + 4c_2 + 8c_3 & = 1 \end{cases}$

$$[A|\vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & -1 & 13 \\ 1 & 2 & 4 & 8 & 1 \end{bmatrix} \to \cdots \to \textbf{rref}[A|\vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So, the linear system has the unique solution $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \\ -1 \end{bmatrix}$ So, the cubic polynomial is $f(t) = 5 - 4t + 3t^2 - t^3$.



perfect fit, but calculation is hard.

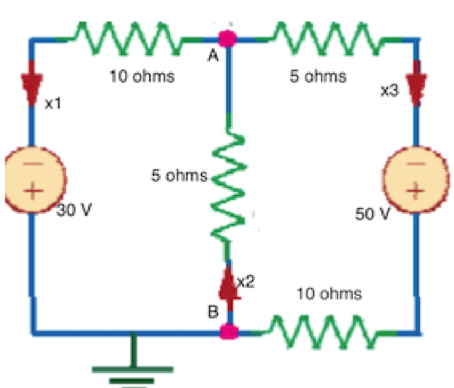
Remark: In statistics, we care more about the linear approximation of the data. We will deal with this in §5.4.

The next circuit example need some background knowledge in physics:

Kirchhoffs current law: At any node, the sum of the currents is zero.

Kirchhoff s voltage law: In any closed loop, the sum of the voltages is zero.

Example 15 (Circuit problem)**.** Find currents in the circuit



There are two nodes and two loops

Node A: $-x_1 + x_2 - x_3 = 0$

Node B: $x_1 - x_2 + x_3 = 0$

Right loop: $5x_2 + 15x_3 = 50$

Left loop: $10x_1 + 5x_2 = 30$