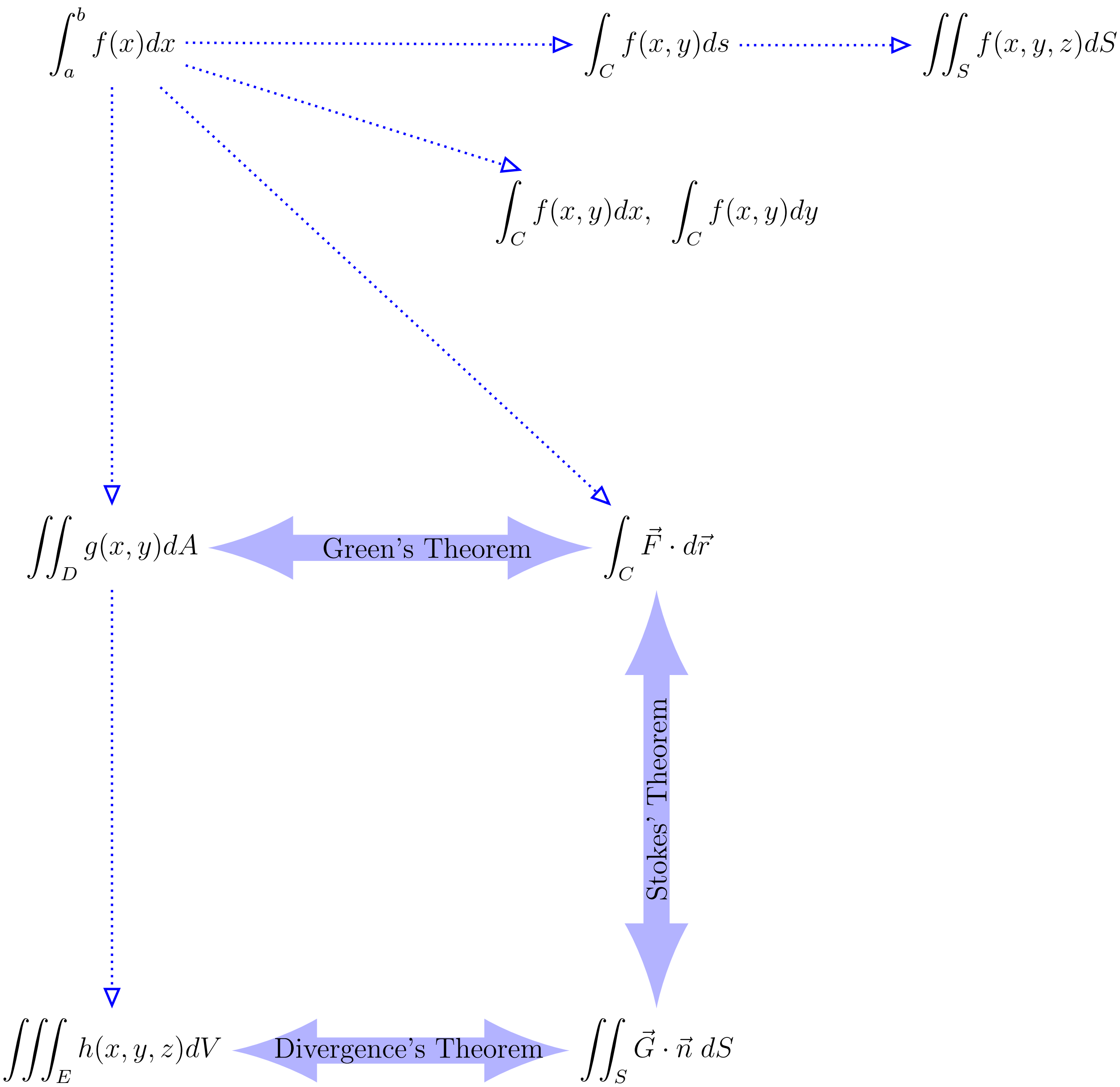
§4 Integration and Vector Fields

This is a quick summary about the structure of Chapter 4 and Chapter 3. It is a overview and we focus on the relationship. It will not cover everything.



The arrows only describe the relations of objects by the theorems. They are not the equality formulas. See the following pages for formulas.

Dotted arrows means generalizations of the concepts.

Let's summarize the computation formulas for line integrals and surface integrals.

§4.2. Line Integrals. Suppose a smooth curve C has the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \leq t \leq b$.

(1) The line integral of $f(x, y)$ (with respect to arc length) along curve C can be evaluated as

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

(2) The line integral of $f(x, y)$ along C **with respect to** x **and** y can be evaluated as

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt, \qquad \int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

(3) Let $\vec{F}$ be a vector field (on $\mathbb{R}^2$ or $\mathbb{R}^3$) defined on a curve C ($\vec{r}(t)$, $a \leq t \leq b$). Let $\vec{T}$ be the unit tangent vector at the point $(x, y, z) \in C$. Then the **line integral of** $\vec{F}$ **along** C is $\int_C \vec{F} \cdot \vec{T} ds = \int_C \vec{F} \cdot d\vec{r}$, which can be computed by

Theorem.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

§4.3 Two Operations on vector fields: Curl and Divergence Let $\vec{F} = \langle P, Q, R\rangle$ be a vector field on $\mathbb{R}^3$. Let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ be a vector operation.

The Curl of F is

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}.$$

The divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Let $\vec{F}$ be a vector field such that P, Q, R have continuous partial derivatives on $\mathbb{R}^3$, then

Theorem 1. $\operatorname{div} \operatorname{curl} \vec{F} = 0$

Theorem 2. $\vec{F}$ is **conservative** if and only if $\operatorname{curl} \vec{F} = \vec{0}$

§4.5 Surface Integrals, Flux.

(1) Suppose a surface S is defined by $\vec{r}(u,v)$. The surface integral can be computed by

$$\iint_S f(x,y,z)dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\; dA$$

If the surface S has an equation $z = g(x,y)$, then $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$, so

$$\iint_S f(x,y,z)dS = \iint\limits_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

(2) Let S be a orientable surface with unit normal vector $\vec{n}$. Let $F = \langle P, Q, R\rangle$ be a vector field. The surface integral (**Flux**) of F over S is $\iint_S \vec{F}\cdot\vec{n}\; dS = \iint_S \vec{F}\cdot d\vec{S}$ (Two notations)

Theorem.

Suppose a surface S is defined by $\vec{r}(u,v)$. The surface integral of a vector field $\vec{F} = \langle P, Q, R\rangle$ over S can also be computed by

$$\iint_S \vec{F}\cdot\vec{n}\; dS = \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\; dA$$

Theorem.

If the surface S is defined by $z = g(x,y)$, then the surface integral of $\vec{F}$ over S can also be computed by

$$\iint_S \vec{F}\cdot\vec{n}\; dS = \iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\; dA$$

Four Big Theorems.

Theorem. Fundamental Theorem for line integrals, §4.3

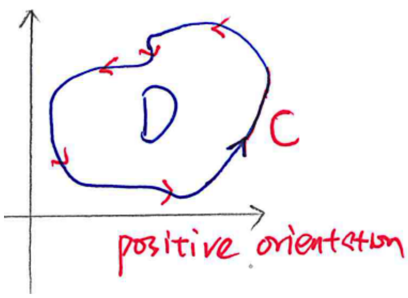
If C is smooth and $\vec{F}$ is **conservative** ($\vec{F} = \nabla f$), then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Theorem. Green's Theorem. §4.4

Let $D \subset \mathbb{R}^2$ be a region bounded by a closed curve C with positive direction.
Suppose C is piecewise-smooth defined by the vector function $\vec{r}(t)$, $a \le t \le b$.
Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ and $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives, then

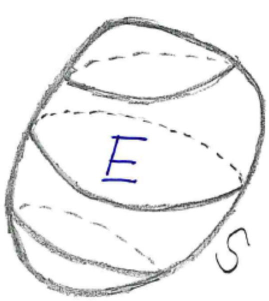
$$\int_C \vec{F} \cdot \vec{r} = \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Theorem. The Divergence Theorem.

Let E be a simple solid region with the boundary surface S. Suppose S has the **positive** orientation. Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field such that P, Q, R have continuous partial derivatives. Then,

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E \nabla \cdot \vec{F}\, dV$$



Theorem. Stokes' Theorem. (§4.7)

Let S be an oriented piecewise-smooth surface with boundary C. Suppose C is a simple, closed, piecewise-smooth boundary curve with positive orientation (counter-clockwise). Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field such that P, Q, R have continuous partial derivatives. Then,

$$\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

For Stokes's Theorem, do not worry to much about the long assumptions.

Look at the following picture and imagine there is water flow $\vec{F}$ passing the surface.