§4.7 Stokes' Theorem

Let S be an oriented piecewise-smooth surface with boundary C.

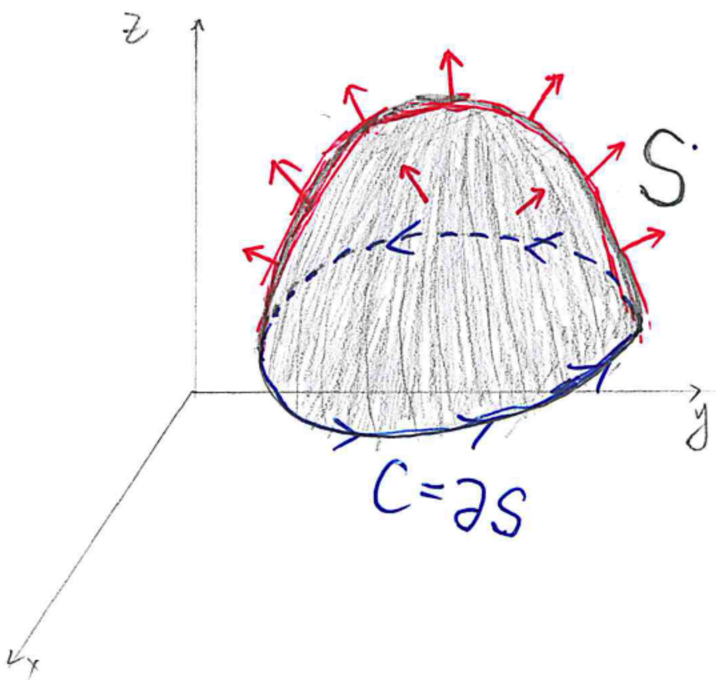
Suppose C is a simple, closed, piecewise-smooth boundary curve with positive orientation (counter-clockwise).

Theorem. Stokes' Theorem.

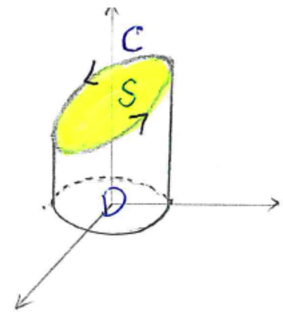
Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field such that P, Q, R have continuous partial derivatives. Then,

$$\int_C \vec{F} \cdot \vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\int_C \vec{F} \cdot \vec{r} = \int_C \vec{F} \cdot \vec{T} ds \overset{\text{Stokes' Theorem}}{=\!=\!=\!=} \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\ dS$$



Example 1. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z - y = 2$ with counterclock orientation.



$$\text{Curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= (1+2y)\vec{k} = \langle 0, 0, 1+2y \rangle$$

Stokes' Thm:

$$\int_C \vec{F} d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$= \iint_D 1+2y \, dA$$

polar system

$$D = \left\{ (r, \theta) \;\middle|\; \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix} \right\}$$

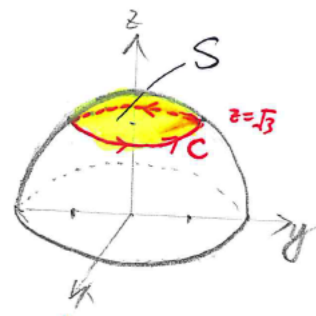
$$= \int_0^{2\pi} \int_0^1 (1+2r\sin\theta)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r + 2r^2 \sin\theta \; dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta \right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta \right) d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta \Big|_0^{2\pi} = \pi$$

Example 2. Compute the integral $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle xz, yz, x^2y \rangle$ and S is the sphere $x^2 + y^2 + z^2 = 4$ above $z = \sqrt{3}$.



C is given by $\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{3} \end{cases}$

that is, $\begin{cases} x^2+y^2=1 \\ z=\sqrt{3} \end{cases}$

vector parametric equation for C: $x=\cos t$, $y=\sin t$, $z=\sqrt{3}$, $0 \leq t \leq 2\pi$

$\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$

Stokes' Thm:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

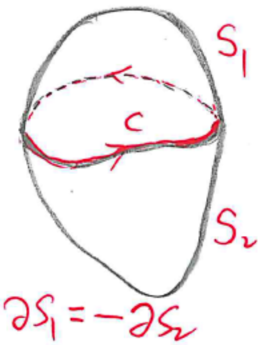
$$= \int_0^{2\pi} -\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t \, dt$$

$$= \int_0^{2\pi} 0 \, dt$$

$$= 0$$

Let S_1 and S_2 be two oriented surfaces with the same oriented boundary curve $C = \partial S_1 = \partial S_2$. Under the hypotheses of Stokes Theorem,

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot \vec{r} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}$$



Let C be an oriented closed curve.

Let $\vec{v}$ be the velocity field in fluid flow.

The line integral $\int_C \vec{v} \cdot d\vec{r}$ computes the tendency of the fluid to move around C.