§4.6 The Divergence Theorem

Recall the vector form of the Green's theorem:

Theorem. Green's Theorem

$$\oint_C \vec{F} \cdot \vec{n}\ ds = \iint_D \operatorname{div} \vec{F}(x, y)\ dA$$

Generalizing to $\mathbb{R}^3$.

Theorem.

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iiint_E \operatorname{div} \vec{F}(x, y, z)\ dV$$

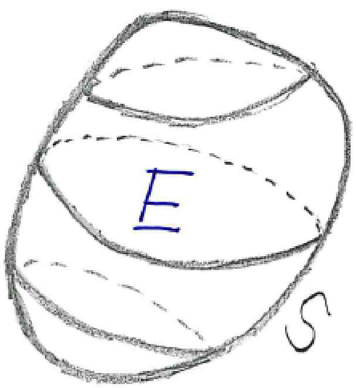
Let E be a simple solid region with the boundary surface S.

Suppose S has the **positive** orientation.

Theorem. The Divergence Theorem.

Let $\vec{F} = \langle P, Q, R\rangle$ be a vector field such that P, Q, R have continuous partial derivatives. Then,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \cdot dV$$



Example 1. (Example4 in §4.5) Find the flux of the vector field $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

- $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$

- Divergence Theorem:

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F}\, dV = \iiint_B 1\, dV = \text{volume of } B$$

$$= \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi.$$

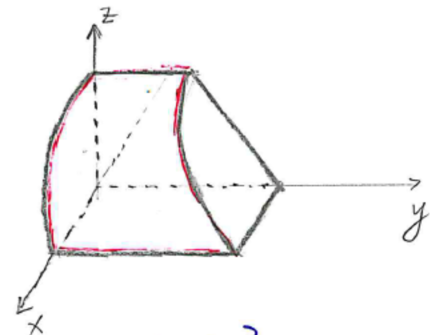
Example 2. Find the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ where

$$\vec{F} = \langle xy, y^2 + \cos(xz^2), e^{xy} \rangle$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and planes $x = 0, y = 0, z = 0$ and $y + z = 2$ in the first octant.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + \cos(xz^2)\right) + \frac{\partial}{\partial z}\left(e^{xy}\right)$$

$$= y + 2y = 3y$$



solid E is $\{(x,y,z) \mid 0 \leq y \leq 2 - z,\ 0 \leq z \leq 1 - x^2,\ 0 \leq x \leq 1\}$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3y\, dV$$

$$= \int_0^1 \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x^2} \frac{3y^2}{2}\Big|_0^{2-z} dz\, dx$$

$$= \int_0^1 \int_0^{1-x^2} \frac{3(2-z)^2}{2}\, dz\, dx$$

$$= \int_0^1 -\frac{(2-z)^3}{2}\Big|_0^{1-x^2} dx$$

$$= -\frac{1}{2}\int_0^1 (x^2+1)^3 - 8\, dx = -\frac{1}{2}\int_0^1 x^6 + 3x^4 + 3x^2 - 7\, dx = \frac{92}{35}$$