§4.5 Flux through a surface – Surface Integrals

Review: From single variable integral to line integrals.

$$\int_a^b f(x)dx$$

Area

Mass

Work

$$\int_C f(x,y)ds \qquad\qquad \int_C \vec{F}\cdot d\vec{r}$$

Flux

Mass

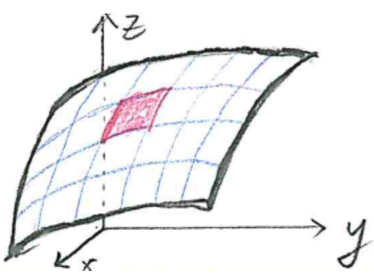
Definition.

We define the **surface integral of** $f(x,y,z)$ **over the surface** S

$$\iint_S f(x,y,z)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

Each area of the patch ΔS_{ij} can by computed by

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v|\Delta u\Delta v.$$



Theorem.

Suppose a surface S is defined by $\vec{r}(u,v)$. The surface integral can be computed by

$$\iint_S f(x,y,z)dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\ dA$$

In particular, if $f = 1$, then

$$\iint_S 1dS = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA = \text{Area of } S$$

Example 1. Compute the surface integral $\iint_S xz\ dS$, where S is the sphere defined by $x^2 + y^2 + z^2 = 1$ in the first octant.

parametric equation : $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$

$0 \le \phi \le \pi/2$

$0 \le \theta \le \pi/2$

$|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$ (By Example 5 in section 3.11)

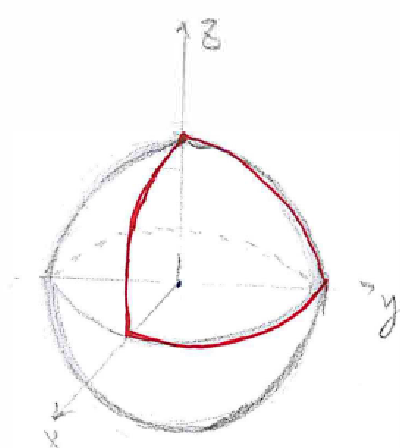
$$\iint_S xz\,dS = \iint_D \sin\phi\cos\phi\cos\theta\,|\vec{r}_\phi \times \vec{r}_\theta|\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin^2\phi\cos\phi\cos\theta\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\,d\theta \int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi$$

$$= \sin\theta\Big|_0^{\pi/2}\ \frac{\sin^3\phi}{3}\Big|_0^{\pi/2}$$

$$= \frac{1}{3}$$



If the surface S has an equation $z = g(x, y)$, then $\vec{r}(x, y) = \langle x, y, g(x, y)\rangle$, so

$$\iint_S f(x,y,z)dS = \iint\limits_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

Similarly, if the surface S has an equation $y = h(x, z)$, then $\vec{r}(x, y) = \langle x, h(x, z), z\rangle$, so

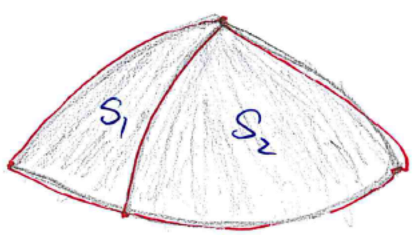
$$\iint_S f(x,y,z)dS = \iint\limits_D f(x,h(x,z),z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\; dA$$

Similarly, if the surface S has an equation $x = h(y, z)$, you can write down formula for the surface integral.

Theorem.

If S is a union of S_1 and S_2, then

$$\iint_S f(x,y,z)dS = \iint_{S_1} f(x,y,z)dS + \iint_{S_2} f(x,y,z)dS$$



Example 2. Evaluate the surface integral $\iint_S x\, dS$, where S is the surface $z = x^2 + y$ for $0 \le x \le 1$ and $0 \le y \le 2$.

$$\frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 1$$

$$\iint_S x\, ds = \iint_D x\sqrt{(2x)^2+1^2+1}\, dA$$

$$= \int_0^2 \int_0^1 x\sqrt{4x^2+2}\, dx\, dy$$

$$= \int_0^2 dy \int_0^1 x\sqrt{4x^2+2}\, dx$$

$$= 2 \cdot \frac{1}{12}\left(4x^2+2\right)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{1}{6}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

$$= \sqrt{6} - \frac{\sqrt{2}}{3}$$

Let $u = 4x^2+2$

$du = 8x\, dx$

$\frac{du}{8} = x\, dx$

$$\int x\sqrt{4x^2+2}\, dx$$

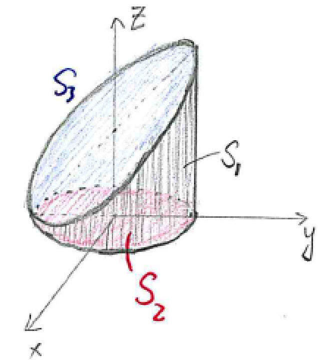
$$= \int u^{\frac{1}{2}} \cdot \frac{1}{8}\, du$$

$$= \frac{1}{8} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{12} u^{\frac{3}{2}} + C$$

$$= \frac{1}{12}\left(4x^2+2\right)^{\frac{3}{2}} + C$$

Example 3. Evaluate the surface integral $\iint_S y\, dS$, where S is the boundary surface of the solid region E enclosed by the cylinder $x^2+y^2=1$, the plane $z=0$ and the plane $z=1+y$.



$$\iint_S y\, dS = \iint_{S_1} y\, dS + \iint_{S_2} y\, dS + \iint_{S_3} y\, dS$$

• (1) S_1 : $x=\cos\theta \quad y=\sin\theta \quad z=z$

$0\le\theta\le 2\pi$

$0\le z\le 1+y = 1+\sin\theta$

$$\vec{r}_\theta\times\vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta,\ \sin\theta,\ 0\rangle$$

$$|\vec{r}_\theta\times\vec{r}_z| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$$

$$\iint_{S_1} y\, dS = \iint_D y\cdot|\vec{r}_\theta\times\vec{r}_z|\, dA = \iint_D y\, dA$$

$$= \int_0^{2\pi}\int_0^{1+\sin\theta} \sin\theta\, dz\, d\theta$$

$$= \int_0^{2\pi} \sin\theta\,(1+\sin\theta)\, d\theta$$

$$= \int_0^{2\pi}\sin\theta\, d\theta + \int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= 0 + \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{2\pi} = \pi$$

(2) S_2 $\quad \underline{x^2+y^2 \leq 1 \quad z=0}$

$x=\cos\theta \quad y=\sin\theta \quad z=z \qquad 0 \leq \theta \leq 2\pi$

$z=0$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$

$|\vec{r}_\theta \times \vec{r}_z| = 1$. $\displaystyle\iint_{S_2} y \, dS = \iint_D y \, dA = \int_0^{2\pi}\int_0^0 \sin\theta \, dz \, d\theta = 0$

(3) S_3 $\quad z = 1+y$

$$\iint_{S_3} y \, dS = \iint_D y \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

$$= \iint_D \sqrt{2} \, y \, dA$$

$D = \{(x,y) \mid x^2+y^2 \leq 1\}$

$= \{(r,\theta) \mid 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\}$

by polar system

$x = r\cos\theta$

$y = r\sin\theta$

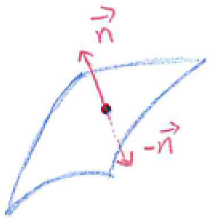
$$= \int_0^{2\pi}\int_0^1 \sqrt{2} \, r\sin\theta \cdot r \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \sin\theta \, d\theta \cdot \int_0^1 r^2 \, dr$$

$$= 0$$

Orientable Surfaces S:

Def: • It is possible to choose a unit normal vector $\vec{n}$ at every point (x, y, z) on S such that $\vec{n}$ varies continuous over S.



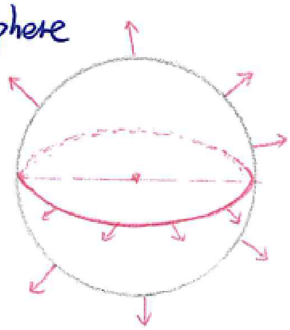
• The choice of $\vec{n}$ provide S an orientation.

• There are two orientations on each orientable surfaces.

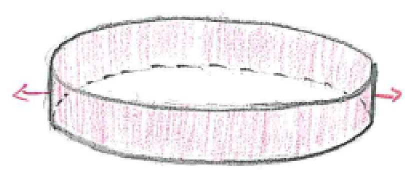
Examples (orientable)

1. sphere

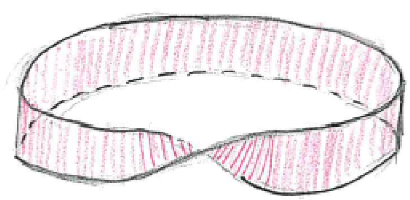
positive orientation



2. cylinder



Example (not orientable)
Möbius strip



Surface integral of vector fields:

Let $\vec{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$.

Definition.

The **surface integral of $\vec{F}$ over** S is defined as

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\ dS$$

This integral is called the **flux of $\vec{F}$ across(through)** S.

Here,

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{and} \quad dS = |\vec{r}_u \times \vec{r}_v| dA.$$

Theorem.

The surface integral of $\vec{F}$ over S can also be computed by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dA$$

Theorem.

If the surface S is defined by $z = g(x, y)$, then the surface integral of $\vec{F}$ over S can also be computed by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\ dA$$

Example 4. Find the surface integral of the vector field $\vec{F} = \langle z, y, x\rangle$ across the sphere $x^2 + y^2 + z^2 = 1$.

parametric equation: $x = \sin\phi\cos\theta \quad y = \sin\phi\sin\theta \quad z = \cos\phi$ $\quad 0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi \rangle$$

$$\vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta)$$

$$= \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r})\cdot(\vec{r}_\phi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta \, d\phi\, d\theta$$

$$= 2\int_0^{\pi}\sin^2\phi\cos\phi\, d\phi \cdot \int_0^{2\pi}\cos\theta\, d\theta + \int_0^{\pi}\sin^3\phi\, d\phi \int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= 2\,\frac{\sin^3\phi}{3}\Big|_0^{\pi} \cdot \sin\theta\Big|_0^{2\pi} + \int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi \cdot \pi$$

$$= 0 + \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Big|_0^{\pi}(\cdot\pi)$$

$$= \frac{4}{3}\pi$$

Example 5. Find the surface integral of the vector field $\vec{F} = \langle x, y, z \rangle$ across the sphere $x^2 + y^2 + z^2 = 4$.

$x = 2\sin\phi\cos\theta \quad y = 2\sin\phi\sin\theta \quad z = 2\cos\phi$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\vec{F}(\vec{r}) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 8\sin^3\phi\cos^2\theta + 8\sin^3\phi\sin^2\theta + 8\sin\phi\cos^2\theta$$

$$= 8\sin^3\phi + 8\sin\phi\cos^2\theta$$

$$\iint_S \vec{F}\, d\vec{S} = \iint_D \vec{F}(\vec{r}) \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} 8\sin^3\phi + 8\sin\phi\cos^2\theta \, d\phi\, d\theta$$

$$= 8\int_0^{2\pi}\sin^3\phi\, d\phi \int_0^{\pi} d\theta + 8\int_0^{2\pi}\sin\phi\, d\phi \int_0^{\pi}\cos^2\theta\, d\theta$$

$$= 8 \cdot \frac{4}{3} \cdot \pi + 0$$

$$= \frac{32}{3}\pi.$$

Example 6. Find the surface integral of the vector field $\vec{F} = \langle x, y, z\rangle$ across the paraboloid $z = -1 + x^2 + y^2$ and the plane $z = 0$.

Denote the paraboloid $z = -1 + x^2 + y^2$ as S_1 and denote the plane $z = 0$ as S_2.

$$\iint_{S_1} \vec{F}\, d\vec{S} = \iint_D \left(-P\cdot\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$= \iint_D \left(-2x^2 - 2y^2 + (-1+x^2+y^2)\right) dA$$

$$= \iint_D (-1-x^2-y^2)\, dA$$

$D:\ x^2+y^2 \le 1$

polar system

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 (-1-r^2)\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 -r-r^3\, dr$$

$$= 2\pi\left(-\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1$$

$$= -\frac{3}{2}\pi$$

$$\iint_{S_2} \vec{F}\, d\vec{S} = \iint_D R\, dA = \iint_D z\, dA = 0$$

$$\iint_S \vec{F}\, d\vec{S} = \iint_{S_1} \vec{F}\, d\vec{S} + \iint_{S_2} \vec{F}\, d\vec{S} = -\frac{3\pi}{2}$$