§4.3-4.4 continue: Curl and Divergence

Two **operations** on vector fields: 1. Curl and 2. Divergence.

1. Curl

Definition.

Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field on $\mathbb{R}^3$. The **curl** of $\vec{F}$ is defined as

$$\operatorname{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Recall the gradient vector

$$\nabla f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}.$$

We can consider ("Del") ∇ as a vector differential operator defined as

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

An easier way to remember the curl of $\vec{F}$ is using cross product

Theorem.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}.$$

Proof.

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\vec{k} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \\
&= \operatorname{curl}(\vec{F})
\end{aligned}$$

Example 1. Find the curl of $\vec{F} = \langle xyz, z^2, 2xy \rangle$

$$\mathrm{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & z^2 & 2xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(z^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial z}(xyz)\right) + \vec{k}\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial y}(xyz)\right)$$

$$= \vec{i}(2x - 2z) - \vec{j}(2y - xy) + \vec{k}(0 - xz)$$

$$= 2(x-z)\,\vec{i} + y(x-2)\,\vec{j} + (-xz)\,\vec{k}$$

or $= \langle 2(x-z),\ y(x-2),\ -xz \rangle$

Theorem.

Let f be a function in $\mathbb{R}^3$ and f has continuous second order partial derivatives, then

$$\mathrm{curl}(\nabla f) = 0$$

If a vector field $\vec{F}$ is conservative, then $\mathrm{curl}(\nabla f) = 0$.

$\vec{F} = \nabla f \qquad \mathrm{curl}(\vec{F}) = \vec{0}$

$$\mathrm{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}(f_{zy} - f_{yz}) - \vec{j}(f_{zx} - f_{xz}) + \vec{k}(f_{yx} - f_{xy})$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$ by Clairaut's Theorem.

Example 2. Whether or not $\vec{F}(x,y,z) = \langle xyz, z^2, 2xy\rangle$ is conservative?

$$\text{Curl}(\vec{F}) = \langle 2(x-z), y(x-2), -xz\rangle \neq \vec{0}$$

So, $\vec{F}$ is not conservative.

Example 3. Whether or not $\vec{F}(x,y,z) = \langle yz, xz, xy + e^z\rangle$ is conservative?

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+e^z \end{vmatrix}$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$$

$$= \langle 0, 0, 0\rangle$$

continuous! partial derivative

So $\vec{F}$ is conservative.

- From §16.2, Example 6, $\vec{F} = \nabla f$ for $f = xyz + e^z + K$

Theorem.

Let $\vec{F}$ be a vector field such that its component functions have continuous partial derivatives and $\text{curl}(\vec{F}) = 0$ on all $\mathbb{R}^3$, then $\vec{F}$ is conservative.

Example 3. continue. Yes

Example 4. (a) Whether or not $\vec{F} = 2xy^3z\vec{i} + 3x^2y^2z\vec{j} + x^2y^3\vec{k}$ is conservative? (b) Find a function f such that $\vec{F} = \nabla f$.

(a) $$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z & 3x^2y^2z & x^2y^3 \end{vmatrix} = \vec{i}(3x^2y^2 - 3x^2y^2) - \vec{j}(2xy^3 - 2xy^3) + \vec{k}(6xy^2z - 6xy^2z)$$

$$= \langle 0, 0, 0 \rangle$$

since all partial derivatives of $\vec{F}$ are continuous, then $\vec{F}$ is conservative.

(b) Find $\vec{f}$ such that $f_x = 2xy^3z$, $f_y = 3x^2y^2z$, $f_z = x^2y^3$

$$\Downarrow$$

$$f = x^2y^3z + g(y,z) \Rightarrow f_y = 3x^2y^2z + g_y \quad f_z = x^2y^3 + g_z$$

$$\Downarrow \qquad \Downarrow$$

$$g_y = 0 \qquad g_z = 0$$

$$g(y,z) = k$$

So $f = x^2y^3z + k$

2. Divergence.

Definition.

The **divergence** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is defined by

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Remark: $\operatorname{div} \vec{F}$ is a function from $\mathbb{R}^3$ to $\mathbb{R}$.

Theorem.

The divergence of $\vec{F}$ can be written as dot product

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

Example 5. Find the divergence of $\vec{F} = xy^2\vec{i} - y^2z\vec{j} + xe^z\vec{k}$.

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(xe^z)$$

$$= y^2 - 2yz + xe^z.$$

Theorem.

Let $\vec{F}$ be a vector field such that its component functions have continuous partial derivatives on $\mathbb{R}^3$, then

$$\text{div}\,\text{curl}\,\vec{F} = 0$$

Proof.

$$\text{div}\,(\text{curl}\,\vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \cdots = 0$$

Clairaut's Theorem

Example 6. Show that $\vec{F} = xy^2\vec{i} - y^2z\vec{j} + xe^z\vec{k}$ can not be the curl of any other vector field.

If $\vec{F} = \text{curl}(\vec{G})$, then $\text{div}(\vec{F}) = \text{div}(\text{curl}(\vec{G})) = 0$.

$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(xe^z)$$

$$= y^2 - 2yz + xe^z$$

$$\neq 0$$ So, $\vec{F}$ is not the curl of any vector field.

Remark: 1. Both Divergence and Curl comes from Physics. Look at(Youbube: 3Blue1Brown Divergence and curl: https://www.youtube.com/watch?v=rB83DpBJQsE). It is helpful for intuition, but it can not replace the calculation in this section.

2. $\text{div}(\nabla f) = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplace operator.

Equation

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

in Green's Theorem can be written in the vector form

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \vec{k}\ dA}$$

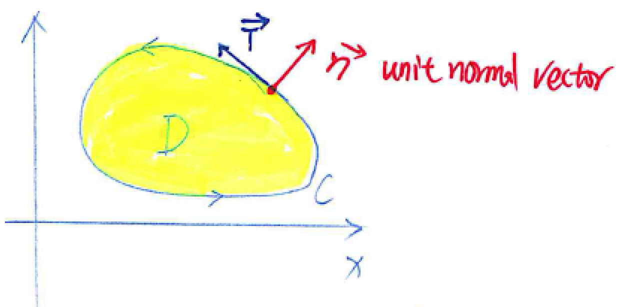
Here $\vec{F} = \langle P, Q, 0 \rangle$.

$$\operatorname{curl}(\vec{F}) = (\qquad)\vec{i} + (\qquad)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

$$\text{so} \quad \operatorname{curl}(\vec{F}) \cdot \vec{k} = 0 + 0 + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Another vector form of the Green's theorem:

$$\boxed{\oint_C \vec{F} \cdot \vec{n}\ ds = \iint_D \operatorname{div} \vec{F}\ dA}$$



$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$

$$\vec{n} = \frac{y'(t)}{|\vec{r}'(t)|} \cdot \vec{i} - \frac{x'(t)}{|\vec{r}'(t)|} \cdot \vec{j}$$

$$\oint_C \vec{F} \cdot \vec{n}\ ds = \int_a^b \vec{F} \cdot \vec{n} \cdot |\vec{r}'(t)|\, dt$$

$$= \int_a^b P(x(t), y(t))\, y'(t) - Q(x(t), y(t))\, x'(t)\, dt$$

$$= \int_C P dy - Q dx = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_D \operatorname{div} \vec{F}\, dA.$$