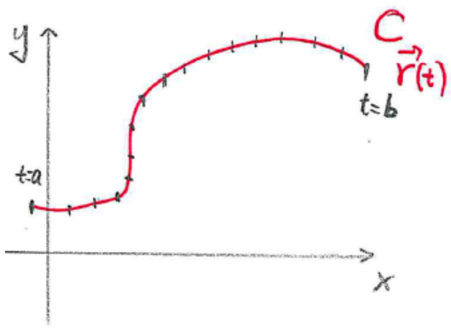
# §4.2 Line integrals

### 1. Line Integral in plane $\mathbb{R}^2$

**Recall:** §1.6 Suppose a smooth curve $C$ has the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \leq t \leq b$.



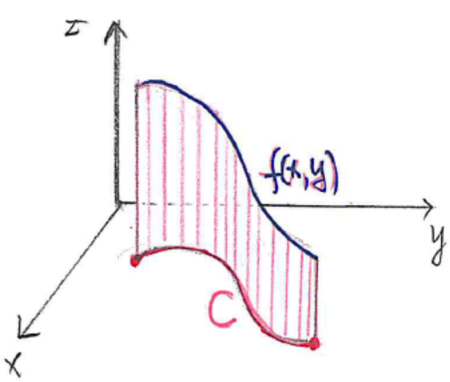

If the curve is traversed exactly once as increases from $a$ to $b$, then its **length** is

$$\begin{aligned} L &= \int_a^b |\vec{r}\,'(t)| dt \\ &= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned}$$

**Definition.**

If $f(x, y)$ is function defined on the curve $C$, then the line integral of $f$ along $C$ is

$$\int_C f(x, y) ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$



**Theorem. Computation.**

The line integral of $f(x,y)$ along curve $C$ can be evaluated as

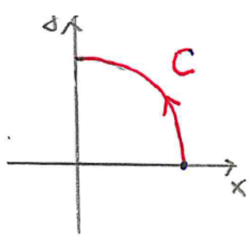
$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

Recall: The arc length function $s(t)$ is the length of the curve between $\vec{r}(a)$ and $\vec{r}(t)$ defined by $s(t) = \int_a^t |\vec{r}\,'(u)|du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}du$

From the Fundamental Theorem of Calculus, differentiate both sides, we have

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

**Example 1.** Evaluate $\int_C (3-xy^2)ds$, where $C$ is the first quadrant of the unit circle $x^2+y^2=1$.




$x = \cos t \qquad x'(t) = -\sin t$

$y = \sin t \qquad y'(t) = \cos t$

$0 \leq t \leq \frac{\pi}{2}$

$$\int_C 3 - xy^2 ds = \int_0^{\frac{\pi}{2}} [3 - (\cos t)(\sin^2 t)] \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} (3 - \sin^2 t \cos t)\, dt$$

$$= 3t - \frac{\sin^3 t}{3} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{2} - \frac{1}{3}$$

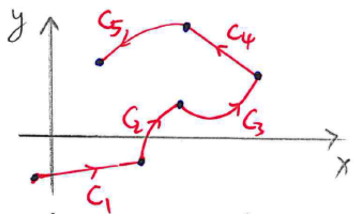
Let $\rho(x,y)$ be the density function on a curve (wire) $C$. Then the **mass** of the wire $C$ is

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)ds$$

The **center of mass** is $(\bar{x}, \bar{y})$ computed by

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Suppose $C$ is a piecewise-smooth curve.

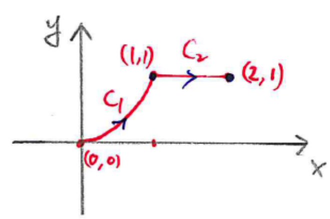



Then,

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds$$

**Example 2.** Evaluate $\int_C 2xds$, where $C$ is the arc $C_1$ of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment $C_2$ from $(1,1)$ to $(2,1)$.

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds.$$




(1.) $C_1$ $\quad x=t$, $y=t^2$, $0 \le t \le 1$

$$\int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{x'(t)^2+y'(t)^2}\,dt$$
$$= \int_0^1 2t\sqrt{1+(2t)^2}\,dt$$
$$= \int_0^1 (1+4t^2)^{\frac{1}{2}}\,dt^2$$
$$= \frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{\frac{3}{2}}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}$$

(2.) $C_2$. $\quad x=t$, $y=1$, $1 \le t \le 2$

$$\int_{C_2} 2x\,ds = \int_1^2 2t\sqrt{1^2+0}\,dt$$
$$= t^2\Big|_1^2 = 4-1 = 3$$

So, $$\int_C 2x\,ds = \frac{5\sqrt{5}-1}{6} + 3$$

**Definition.**

The line integral of $f$ along $C$ **with respect to** $x$ is

$$\int_C f(x,y)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

The line integral of $f$ along $C$ **with respect to** $y$ is

$$\int_C f(x,y)dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

Suppose a smooth curve $C$ has the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$. The line integral of $f$ along $C$ **with respect to** $x$ **and** $y$ can be evaluated as

**Theorem.**

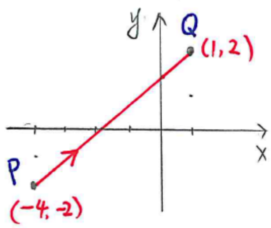
$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$
$$\int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

$\int_C f(x,y)ds$ will be called the line integral of $f$ along $C$ **with respect to arc length**.

Notation:

$$\int_C f(x,y)dx + g(x,y)dy := \int_C f(x,y)dx + \int_C g(x,y)dy$$

**Example 3.** Evaluate $\int_C y^2dx - 2xdy$, where $C$ is the line segment from $(-4,-2)$ to $(1,2)$.




position vector $\vec{r_0} = \langle -4, -2 \rangle$

direction vector $\vec{v} = \vec{PQ} = \langle 5, 4 \rangle$

Line Segment $C$:

$\vec{r}(t) = \vec{r_0} + t\vec{v} = \langle -4, -2 \rangle + t \langle 5, 4 \rangle$

$x = -4 + 5t$ $\quad$ $0 \leq t \leq 1$ $\quad$ $dx = 5\,dt$

$y = -2 + 4t$ $\quad\quad\quad\quad$ $dy = 4\,dt$

$$\int_C y^2 dx - 2x\,dy = \int_0^1 (-2+4t)^2\, 5\,dt - 2(-4+5t)\,4\,dt$$

$$= \int_0^1 5(16t^2 - 24t + 4) + 32\,dt$$

$$= 5\left(\frac{16}{3}t^3 - 12t^2 + 4t\right) + 32t \Big|_0^1$$

$$= \frac{56}{3}$$

**Example 4.** Evaluate $\displaystyle\int_C y^2 dx$, where $C$ is the arc of the parabola $x = 2 - y^2$ from $(1, -1)$ to $(-2, 2)$.

$x = 2 - t^2$, $y = t$ and $-1 \leq t \leq 2$.

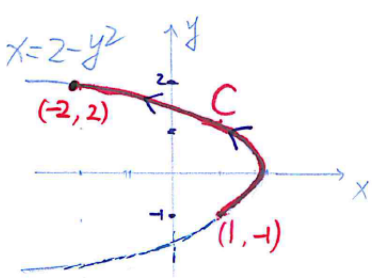
$$\int_C y^2\,dx = \int_{-1}^2 t^2\, x'(t)\,dt$$

$$= \int_{-1}^2 t^2(-2t)\,dt$$

$$= \int_{-1}^2 -2t^3\,dt$$

$$= -2\frac{t^4}{4}\Big|_{-1}^2$$

$$= -\frac{15}{2}$$



## 2. Line integral in space $\mathbb{R}^3$.

Suppose a smooth curve $C$ has the vector equation $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$.

**Definition.**

The line integral of $f$ along $C$ **with respect to the arc length** is

$$\int_C f(x, y, z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

The line integral of $f$ along $C$ **with respect to** $z$ is

$$\int_C f(x, y, z)dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i$$

**Theorem.**

The line integral of $f$ along $C$ with respect to the arc length can be evaluated as

$$\int_C f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

The line integral of $f$ along $C$ with respect to $z$ can be evaluated as

$$\int_C f(x, y, z)dz = \int_a^b f(x(t), y(t), z(t))z'(t)dt$$

**Example 5.** Evaluate $\int_C 2x \sin z \, ds$, where $C$ is the helix defined by $x = \sin t$, $y = \cos t$, $z = t$ for $0 \le t \le \pi$.

$$\int_C 2x \sin z \, ds$$

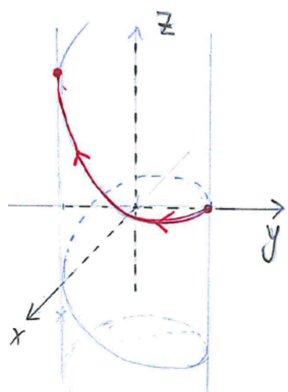
$$= \int_0^{\pi} 2 \sin t \sin t \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

$$= \int_0^{\pi} 2 \sin^2 t \sqrt{\cos^2 t + \sin^2 t + 1} \quad dt$$

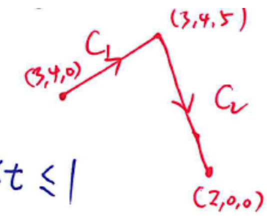
$$= \sqrt{2} \int_0^{\pi} 1 - \cos 2t \quad dt$$

$$= \sqrt{2} \left( t - \frac{1}{2} \sin 2t \right) \Big|_0^{\pi}$$

$$= \sqrt{2}\,\pi$$



**Example 6.** Evaluate $\int_C y dx + z dy + x dz$, where $C$ is the union of the line segment $C_1$ from $(3,4,0)$ to $(3,4,5)$ and the line segment $C_2$ from $(3,4,5)$ to $(2,0,0)$.




For $C_1$: $\vec{r}(t) = \langle 3, 4, 0\rangle + t\langle 0, 0, 5\rangle$

$x = 3 \quad y = 4 \quad z = 5t \qquad 0 \le t \le 1$

$$\int_{C_1} y dx + z dy + x dz = \int_0^1 4\,(0) + 5t\,(0) + 3\,(5)\, dt = 15$$

For $C_2$: $\vec{r}(t) = \langle 3, 4, 5\rangle + t\langle -1, -4, -5\rangle$

$x = 3 - t \quad y = 4 - 4t \quad z = 5 - 5t \qquad 0 \le t \le 1$

$$\int_{C_2} y dx + z dy + x dz = \int_0^1 (4-4t)(-1)\, dt + (5-5t)(-4)\, dt + (3-t)(-5)\, dt$$

$$= \int_0^1 29t - 39\, dt$$

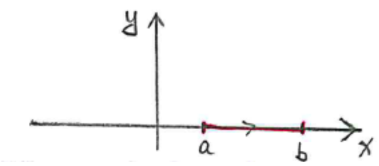
$$= \frac{29t}{2} - 39t \Big|_0^1 = -24.5$$

So, $\int_C y dx + z dy + x dz = 15 - 24.5 = -9.5$

### 3. Line Integrals of Vector Fields.

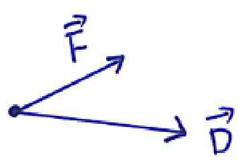
**Recall Calculus 1.** The work done by a force function $f(x)$ in moving a particle from $a$ to $b$ along $x$-axis is

$$W = \int_a^b f(x)dx.$$

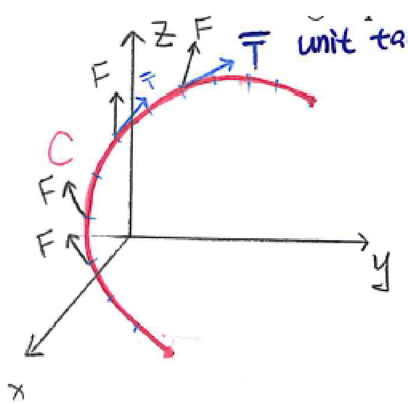



**Recall §1** The work done by a constant force $\vec{F}$ along displacement vector $\vec{D}$ is given by

$$W = \vec{F} \cdot \vec{D}$$




**Question:** How to calculate the work done by a force function $\vec{F}(x, y, z)$ moving a particle along a curve $C$?




$\vec{T}$ unit tangent vector

$$\vec{F}(x_i^*, y_i^*, z_i^*) \bullet \left[(\Delta S_i)\, \vec{T}(t_i^*)\right]$$

$$\sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \bullet \vec{T}(t_i^*)\ \Delta S_i$$

$$W = \int_C \vec{F} \bullet \vec{T}\, ds$$

**Definition.**

Let $\vec{F}$ be a vector field (on $\mathbb{R}^2$ or $\mathbb{R}^3$) defined on a curve $C$ ($\vec{r}(t)$, $a \leq t \leq b$). Then the **line integral of $\vec{F}$ along $C$** is

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{T}$ is the unit tangent vector at the point $(x, y, z) \in C$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \qquad ds = |r'(t)|\, dt$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \cdot |r'(t)|\, dt$$

$$= \int_C \vec{F} \cdot \vec{r}\,'(t)\, dt.$$

- If $\vec{F} = \langle \vec{P}, \vec{Q}, \vec{R} \rangle$ , $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

then $\boxed{\int_C \vec{F} \cdot \vec{T}\, ds} = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$

$$= \int_a^b (\vec{P}) x'(t) + (\vec{Q}) \cdot y'(t) + (\vec{R}) z'(t)\, dt.$$

$$\boxed{= \int_C P\, dx + Q\, dy + R\, dz}.$$

**Example 7.** Find the work done by a force field $\vec{F}(x, y) = \langle y^2, -xy\rangle$ moving a particle along the curve $C$ given by $\vec{r}(t) = \langle \sin t, \cos t\rangle$, when $0 \le t \le \pi/2$.

$$\int_C \vec{F}\, d\vec{r} = \int_0^{\frac{\pi}{2}} \langle y^2, -xy\rangle \cdot \vec{r}\,'(t)\, dt$$

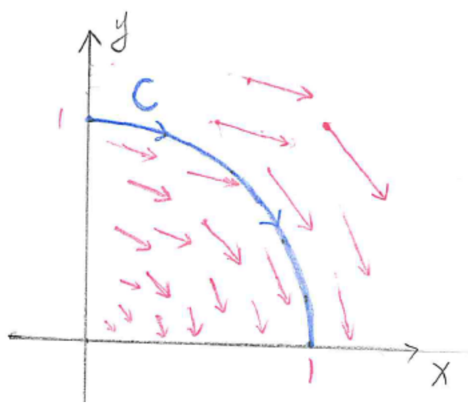
$$= \int_0^{\frac{\pi}{2}} \langle \cos^2 t, -\sin t \cos t\rangle \cdot \langle \cos t, -\sin t\rangle\, dt$$

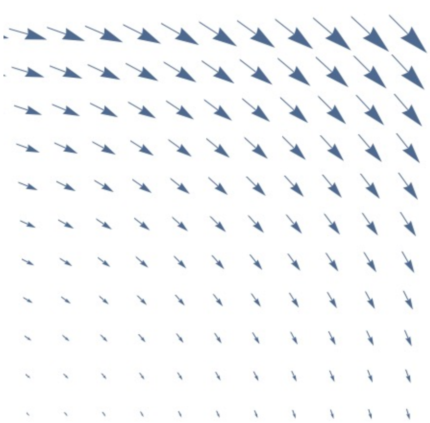
$$= \int_0^{\frac{\pi}{2}} \cos^3 t + \sin^2 t \cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos t\, dt$$

$$= \sin t \Big|_0^{\frac{\pi}{2}}$$

$$= 1$$



**Example 8.** Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ and $C$ is given by $x = t$, $y = t^2$, $z = t^3$ for $0 \leq t \leq 1$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \vec{r}'(t) \cdot dt$$

$$= \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 t^3 + 5t^6 \, dt$$

$$= \frac{t^4}{4} + \frac{5t^7}{7} \Big|_0^1$$

$$= \frac{27}{28}$$