§4.1 Vector Fields

Recall:

Definition.

The **gradient** of a function $f(x, y)$ is the vector function ∇f by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j}$$

The **gradient** of a function $f(x, y, z)$ is the vector function ∇f by

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k}$$

Example 1. Find the gradient of $f(x, y) = \sin(xy) + e^y$.

$$\nabla f = \langle f_x, f_y\rangle = \langle y\cos(xy), x\cos(xy) + e^y\rangle$$

Example 2. Find the gradient of $f(x, y, z) = xyz$.

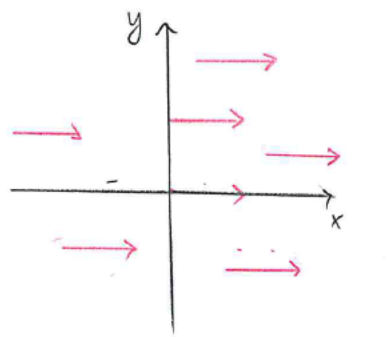
$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle yz, xz, xy\rangle$$

Definition.

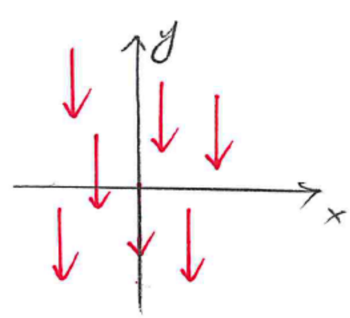
A **vector field** on $D \subset \mathbb{R}^2$ is a vector function $\vec{F}(x, y)$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\vec{F}(x, y)$.

Example 3.

$\vec{F}(x, y) = \langle 1, 0\rangle$

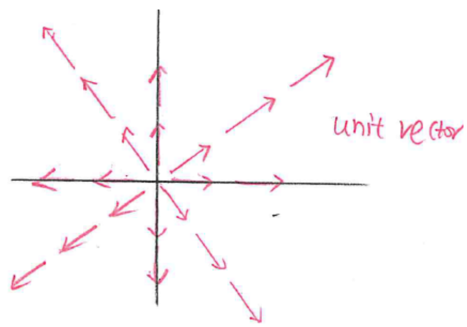


$\vec{G}(x, y) = \langle 0, -1\rangle$.

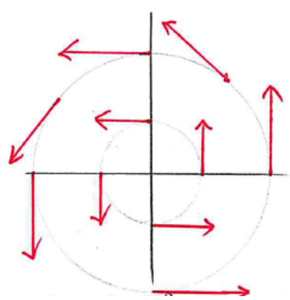


Example 4. $\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

$\vec{F}(x,y) = \nabla(\frac{1}{2}\sqrt{x^2+y^2})$



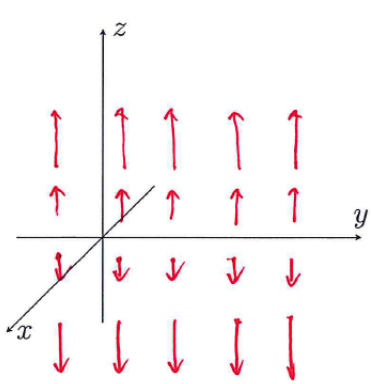
Example 5. $\vec{G}(x,y) = -y \cdot \vec{i} + x \cdot \vec{j}$



Definition.

A **vector field** on $E \subset \mathbb{R}^3$ is a function $\vec{F}(x,y,z)$ that assigns to each point $(x,y,z) \in E$ a three-dimensional vector $\vec{F}(x,y,z)$.
Similar definition works for $\mathbb{R}^3$ and $\mathbb{R}^n$.

Example 6. Sketch the vector field on $\mathbb{R}^3$ given by $\vec{F}(x,y,z) = \langle 0,0,z \rangle$.



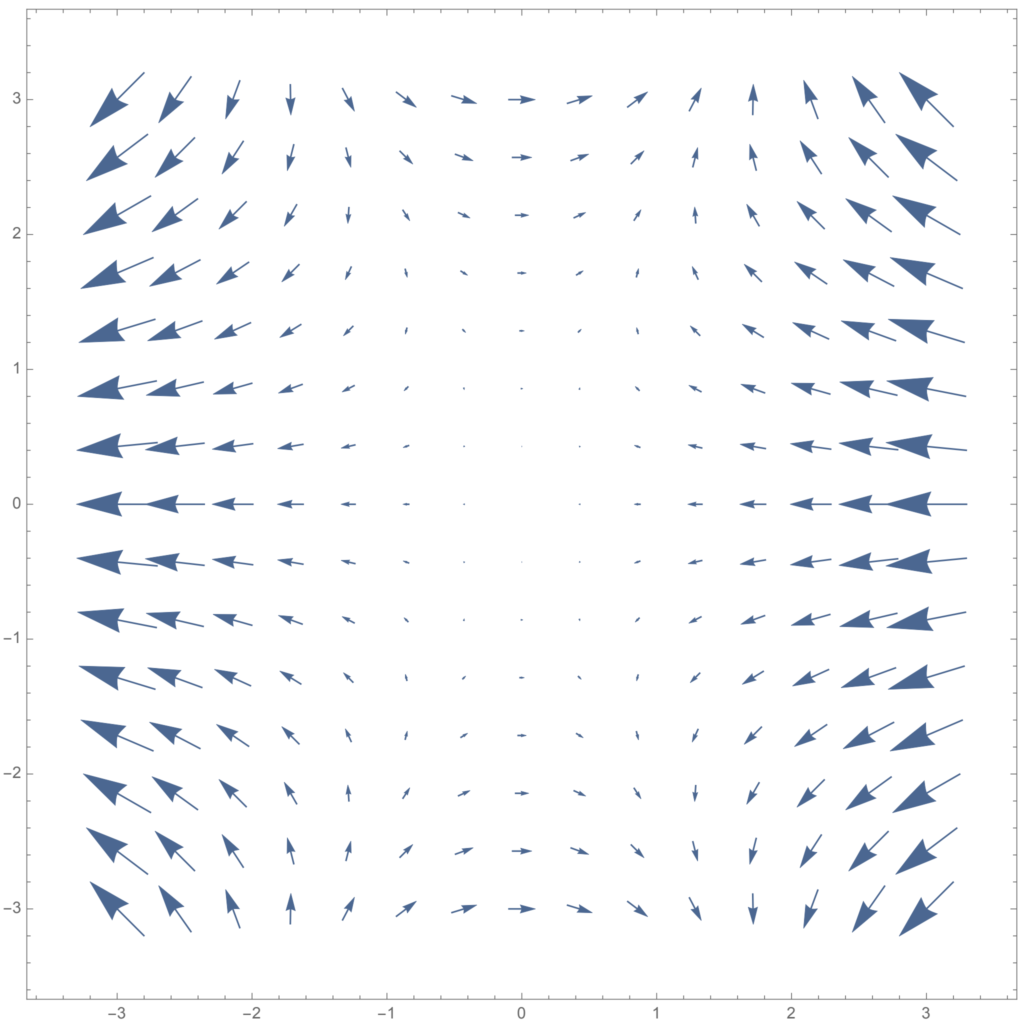
Gradient vector fields

Definition.

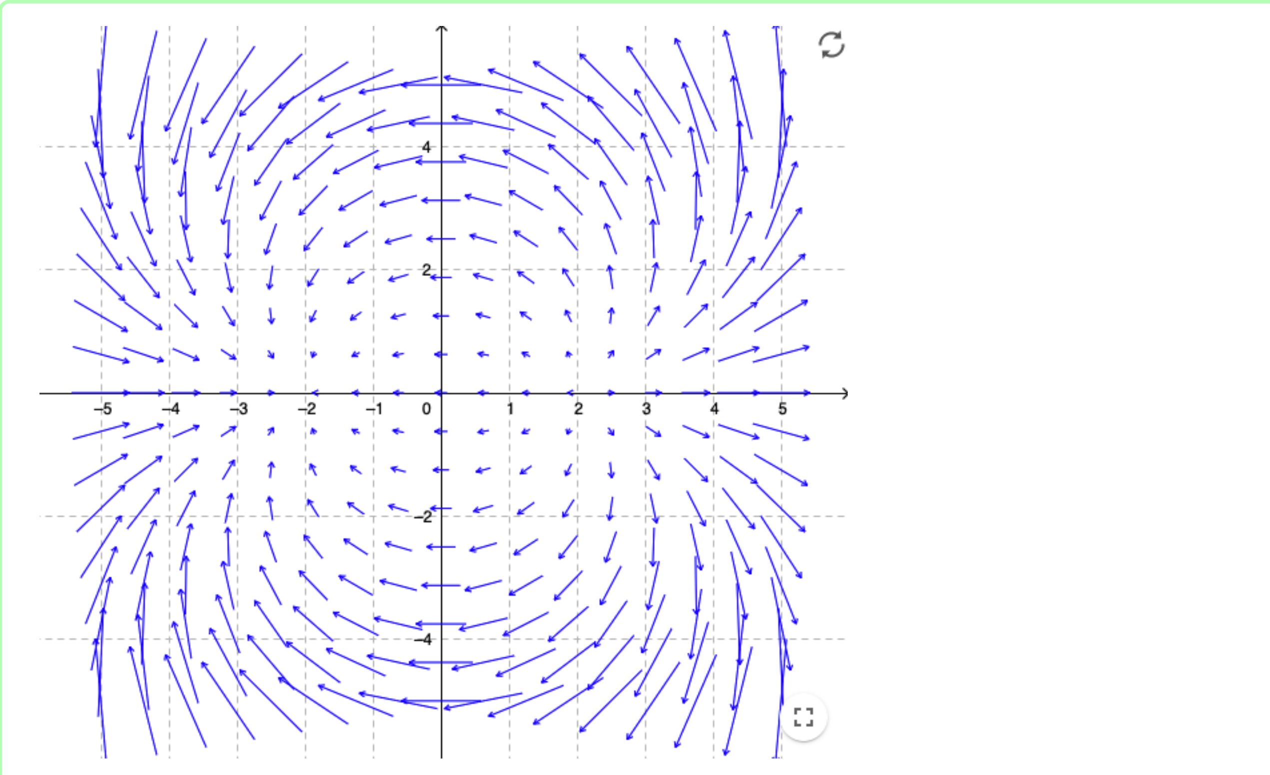
A vector field $\vec{F}$ is called a **conservative vector field** if it is the gradient of some function f, that is, $\vec{F} = \nabla f$.
f is called a **potential function** for $\vec{F}$.

Example 7. Find the gradient vector field of $f(x, y) = y^2x - x^3$.

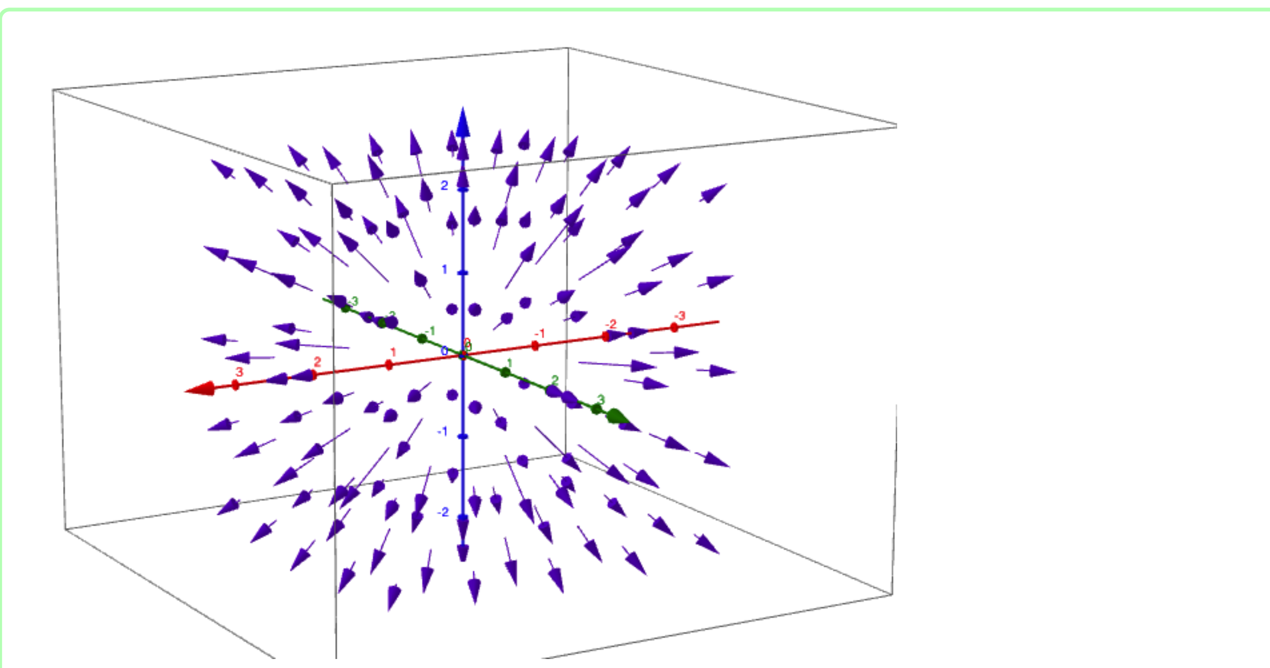
$$\nabla f = \langle f_x, f_y \rangle = \langle y^2 - 3x^2, 2xy \rangle$$



Example 8. $\vec{F} = \langle x^2 - y^2 - 4, 2xy \rangle$



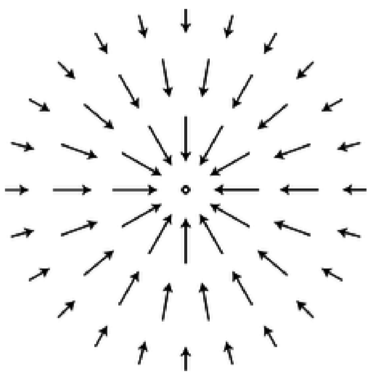
Example 9. $\vec{F} = \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \rangle$



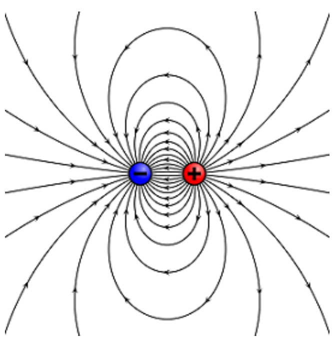
Example 10. (Gravitational Field) Let $\vec{x} = \langle x, y, z \rangle \in \mathbb{R}^3$. The gravitational force acting on the object at $\vec{x}$ is

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$

m and M are masses of the two objects. $G = 6.67408 \times 10^{-11}$ is the universal Gravitational constant.



Example 11. (Electric Field) Electric Force exerted by an electric charge Q at an point $\vec{x} = (x, y, z)$ is $\vec{F}(\vec{x}) = \frac{\epsilon q Q}{|\vec{x}|^3}\vec{x}$



Example 12. Magnetic Field