§3.8 Density and mass

In calculus 1, we can use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. Now, we can deal with **variable density**.

1. Mass of a plate

Suppose the lamina occupies a region D of the xy-plane and its **density** at a point $(x, y) \in D$ given by $\rho(x, y)$. Then the **mass** m of the lamina can be computed by

$$\boxed{m = \iint_D \rho(x, y)dA}$$

Reason:

$$m = \lim_{\Delta A \to 0} \sum_{i,j} \rho(x^*_{i,j}, y^*_{i,j})\Delta A = \iint_D \rho(x, y)dA$$

Example 1. Find the **mass** of a triangular plate with vertices $(0, 1)$, $(1, 0)$, $(1, 1)$ such that the density given by $\rho(x, y) = 2xy$

$$m = \text{Mass} = \iint_D \rho(x,y)\,dA$$

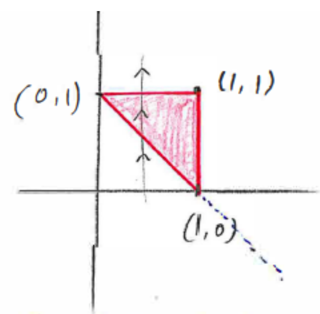
$$= \int_0^1 \int_{1-x}^1 2xy\,dy\,dx$$

$$= \int_0^1 xy^2 \Big|_{1-x}^1 dx$$

$$= \int_0^1 x - x(1-x)^2\,dx$$

$$= \int_0^1 2x^2 - x^3\,dx$$

$$= \frac{2x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$



formula for the line:

direction vector $\langle 1, -1 \rangle$

$$\frac{x-1}{1} = \frac{y-0}{-1}$$

$$x-1 = -y$$

$$y = 1-x$$

2. Mass of a solid

Suppose a solid object occupies a region E in $\mathbb{R}^3$ and its **density** at a point $(x, y, z) \in E$ given by $\rho(x, y, z)$. Then the **mass** m of the solid can be computed by

$$\boxed{m = \iiint_E \rho(x, y, z) dV}$$

Reason:

$$m = \lim_{\Delta V \to 0} \sum_{i,j,k} \rho(x^*_{i,j,k}, y^*_{i,j,k}, z^*_{i,j,k}) \Delta V = \iiint_E \rho(x, y, z) dV$$

Example 2. Let E be the solid region under the plane where $z = y$ and above the rectangle in the xy-plane given by $0 \leq x \leq 2$ and $0 \leq y \leq 1$. Suppose the density δ at each point is given by $\delta(x, y, z) = e^z + xy$ kg/m^3. Find the mass of E.

$$\text{mass} = \int_0^2 \int_0^1 \int_0^y (e^z + xy)\, dz\, dy\, dx =$$

$$\int_0^2 \int_0^1 (e^z + xyz) \Big|_{z=0}^{z=y} dy\, dx = \int_0^2 \int_0^1 \left(e^y + xy^2 - 1\right) dy\, dx =$$

$$\int_0^2 \left(e^y + \frac{xy^3}{3} - y\right) \Big|_{y=0}^{y=1} dx = \int_0^2 \left(e + \frac{x}{3} - 2\right) dx =$$

$$(e - 2)x + \frac{x^2}{6} \Big|_0^2 = 2(e - 2) + \frac{2}{3} \approx 2.10 \text{ kg.}$$

§3.9. Monents and Centers of Mass (Not required)

The **moment about the x-axis** is

$$M_x = \iint_D y\rho(x,y)dA$$

The **moment about the y-axis** is

$$M_y = \iint_D x\rho(x,y)dA$$

(Continue with Example 1.) The **center of mass of a lamina** is

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m})$$

where m, M_x and M_y are computed above.

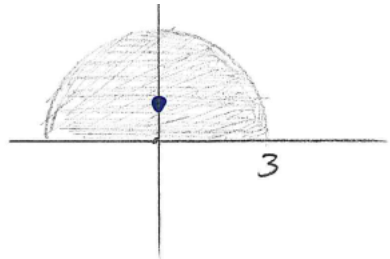
Example 3. Find the **center of mass** of a triangular plate with vertices $(0,1)$, $(1,0)$, $(1,1)$ such that the density given by $\rho(x,y) = 2xy$

$$M_x = \iint_D y\,\rho(x,y)\,dA$$
$$= \int_0^1 \int_{1-x}^1 2xy^2\,dy\,dx$$
$$= \int_0^1 2x \frac{y^3}{3}\Big|_{1-x}^1 dx$$
$$= \int_0^1 \frac{2x}{3} - \frac{2x}{3}(1-x^3)\,dx$$
$$= \int_0^1 2x^2 - 2x^3 + \frac{2}{3}x^4\,dx$$
$$= \frac{2}{3} - \frac{2}{4} + \frac{2}{15} = \frac{3}{10}$$

$$M_y = \iint_D x\,\rho(x,y)\,dA$$
$$= \int_0^1 \int_{1-x}^1 2x^2y\,dy\,dx$$
$$= \int_0^1 x^2y^2\Big|_{1-x}^1 dx$$
$$= \int_0^1 2x^3 - x^4\,dx$$
$$= \frac{3}{10}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{3/10}{5/12}, \frac{3/10}{5/12}\right) = \left(\frac{18}{25}, \frac{18}{25}\right)$$

Example 4. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle $x^2 + y^2 = 3^2$. Find the center of mass of the lamina.



$$f(x,y) = k\sqrt{x^2+y^2}$$

$$0 \le r \le 3$$
$$0 \le \theta \le \pi$$

$$m = \iint_D f(x,y)\, dA$$
$$= \int_0^{\pi} \int_0^3 k\sqrt{r^2}\; r\,dr\,d\theta$$
$$= \int_0^{\pi} \int_0^3 kr^2\,dr\,d\theta$$
$$= \int_0^{\pi} \frac{kr^3}{3}\Big|_0^3\, d\theta$$
$$= \int_0^{\pi} 9k\,d\theta$$
$$= 9k\pi$$

$$M_y = \iint_D x\, f(x,y)\, dA$$
$$= \int_0^{\pi} \int_0^3 kr^3\cos\theta\,dr\,d\theta$$
$$= k\int_0^{\pi} \cos\theta\,d\theta \int_0^3 r^3\,dr$$
$$= 0$$

$$M_x = \iint_D \underline{y}\; \underline{f(x,y)}\, dA$$
$$= \int_0^{\pi}\int_0^3 \underline{r\sin\theta}\; \underline{kr}\; \underline{r\,dr}\,d\theta$$
$$= \int_0^{\pi}\int_0^3 kr^3\sin\theta\,dr\,d\theta$$
$$= \int_0^{\pi} k\sin\theta\,d\theta \int_0^3 r^3\,dr = \left(-k\cos\theta\Big|_0^{\pi}\right)\cdot\left(\frac{r^4}{4}\Big|_0^3\right) = (2k)\left(\frac{81}{4}\right) = \frac{81k}{2}$$

$$\boxed{\begin{aligned}(\bar{x},\bar{y}) &= \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \\ &= \left(0, \frac{9}{2\pi}\right) \\ &\approx (0,\ 1.4324)\end{aligned}}$$