§3.6 Cylindrical and spherical coordinates

1. Triple Integrals in Cylindrical Coordinates

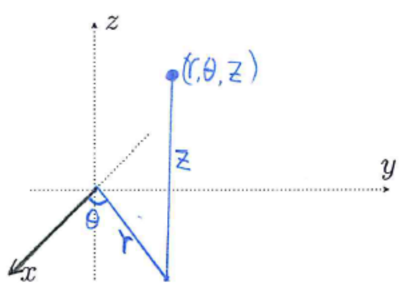
Relation between cylindrical coordinate system (r, θ, z) and rectangular coordinate system (x, y, z).

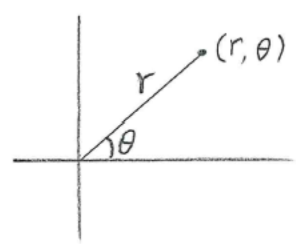
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

(This means for xy-plane we use polar system and keep z.)

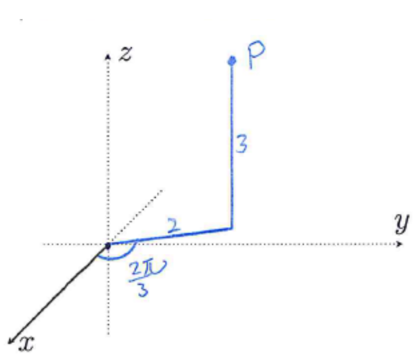
We can obtain formulas converting from rectangular coordinates to cylindrical coordinates:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z$$





Example 1. Find the rectangular coordinates of the point P with cylindrical coordinates $(2, \frac{2\pi}{3}, 3)$.



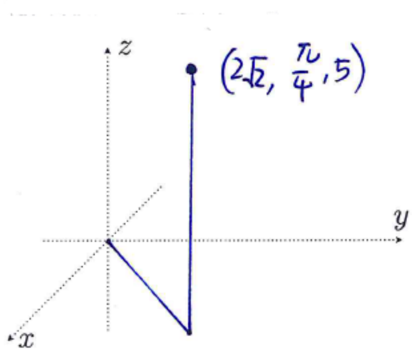
$x = r\cos\theta = 2\cos(\frac{2\pi}{3}) = 2(-\frac{1}{2}) = -1$

$y = r\sin\theta = 2\sin(\frac{2\pi}{3}) = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$

$z = 3$

$(-1, \sqrt{3}, 3)$

Example 2. Find cylindrical coordinates of the point with rectangular coordinates $(2, 2, 5)$.



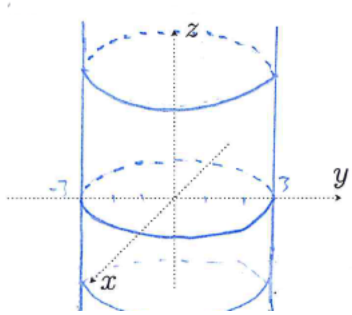
$\tan\theta = \frac{y}{x} = \frac{2}{2} = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

$r^2 = x^2 + y^2 = 4 + 4 = 8$

$\Rightarrow r = 2\sqrt{2}$

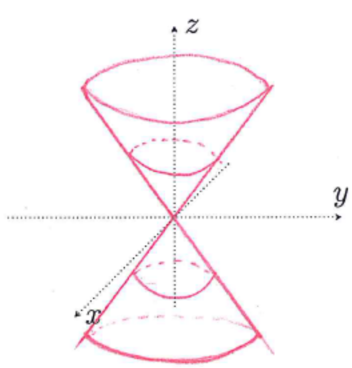
Example 3. Describe the surface with equation in cylindrical coordinates is $r = 3$.



$r = 3$

$\sqrt{x^2 + y^2} = 3$

Example 4. Describe the surface with equation in cylindrical coordinates is $z = r$.



$z = r = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

If a solid region E lies between the graphs of two functions:

$$E = \{(x, y, z) \mid (x, y) \in D,\ g_1(x, y) \le z \le g_2(x, y)\},$$

we already know that

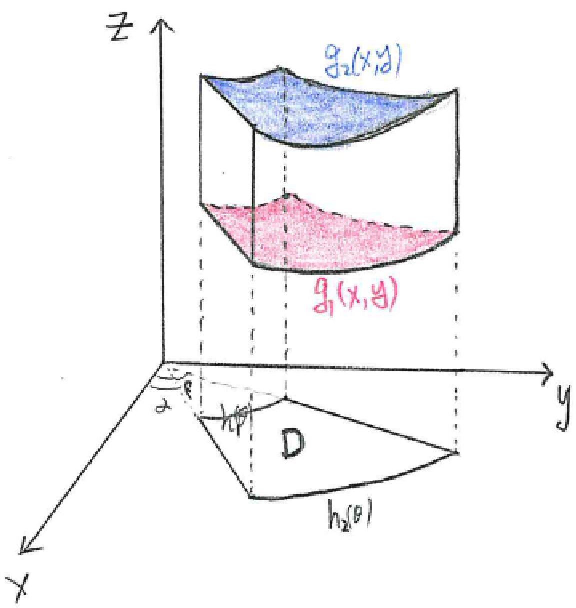
$$\boxed{\iiint_E f(x, y, z)\ dV = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dz\right] dA.}$$

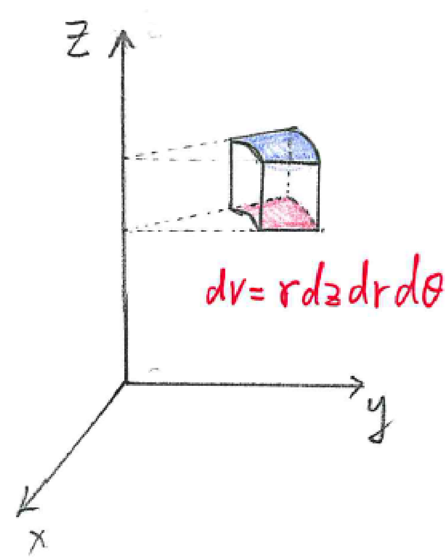
If D can be written as polar coordinates by

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\},$$

then we have the **formula for triple integration in cylindrical coordinates:**

$$\boxed{\iiint_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r\cos\theta, r\sin\theta)}^{g_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dzdrd\theta.}$$



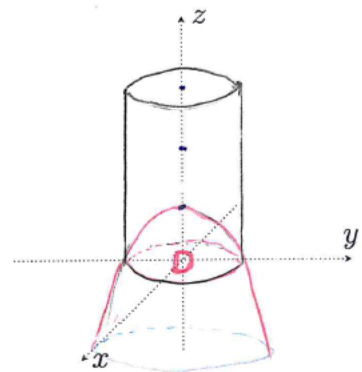


Example 5. A solid E lies in the cylinder $x^2 + y^2 = 1$ above the paraboloid $z = 1 - x^2 - y^2$ and below $z = 3$. The density at point (x, y) is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$. Find the mass of E.

$$1 - x^2 - y^2 \le z \le 3$$

$$1 - r^2 \le z \le 3$$

$$D = \{(r, \theta) \mid 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}$$



$$m = \iiint_E \rho(x, y, z)\, dV$$

$$= \iint_D \left(\int_{1-r^2}^{3} \sqrt{x^2+y^2}\, dz \right) dA$$

$$= \iint_D \left(\int_{1-r^2}^{3} r\, dz \right) dA$$

$$= \iint_D r z \Big|_{1-r^2}^{3} dA$$

$$= \iint_D r(2 + r^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (2r + r^3)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2r^3}{3} + \frac{r^4}{5} \Big|_0^1 d\theta$$

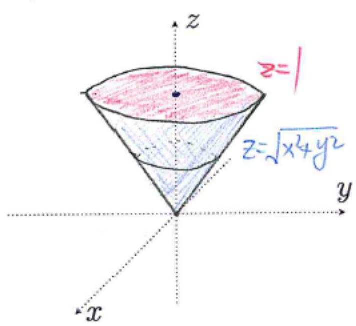
$$= \int_0^{2\pi} \frac{13}{15}\, d\theta = \frac{26}{15}\pi$$

or $m = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^{3} r \cdot r\, dz\, dr\, d\theta$

Example 6. Evaluate $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1}(x^2+y^2)dzdydx$

In cylindrical coordinates

$E=\{(r,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le 1\}$



$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1}(x^2+y^2)dz\,dy\,dx$

$\sqrt{x^2+y^2}\le z\le 1$

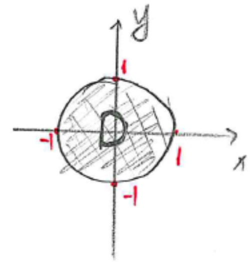
$=\iiint_E (x^2+y^2)\,dV$

$-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$

$\Rightarrow y^2\le 1-x^2$

$\Rightarrow x^2+y^2\le 1$

$=\int_0^{2\pi}\int_0^1\left(\int_r^1 r^2\cdot r\,dz\right)dr\,d\theta$



$=\int_0^{2\pi}\int_0^1 r^3(1-r)\,dr\,d\theta$

$=\int_0^{2\pi}d\theta\cdot\int_0^1 r^3-r^4\,dr$

$=2\pi\left(\frac{r^4}{4}-\frac{r^5}{5}\right)\Big|_0^1$

$=2\pi\left(\frac{1}{20}\right)$

$=\frac{\pi}{10}$

Example 7. Evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

$$=\iiint_E x^2+y^2\,dV$$

$$=\int_0^{\pi}\int_0^2\int_r^2 r^2\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{\pi}\int_0^2 r^3(2-r)\,dr\,d\theta$$

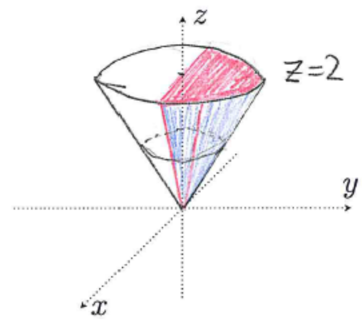
$$=\int_0^{\pi}\int_0^2 2r^3-r^4\,dr\,d\theta$$

$$=\int_0^{\pi}\frac{2r^4}{4}-\frac{r^5}{5}\Big|_0^2\,d\theta$$

$$=\int_0^{\pi}\frac{8}{5}\,d\theta$$

$$=\frac{8}{5}\theta\Big|_0^{\pi}$$

$$=\frac{8\pi}{5}$$

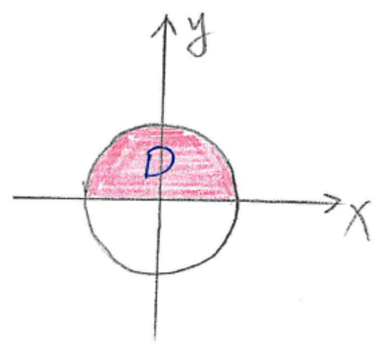


$$\sqrt{x^2+y^2}\leq z\leq 2$$

$$0\leq y\leq\sqrt{4-x^2}$$

$$y^2\leq 4-x^2$$

$$x^2+y^2\leq 4$$



2. Triple Integrals in Spherical Coordinates

Relation between spherical coordinate system (ρ, θ, ϕ) and rectangular coordinate system (x, y, z).

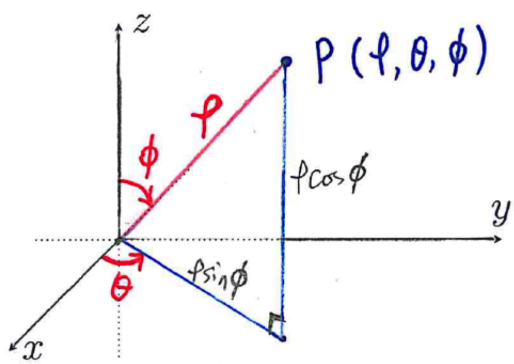
$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi$$

Here, $\rho \geq 0$, $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$.

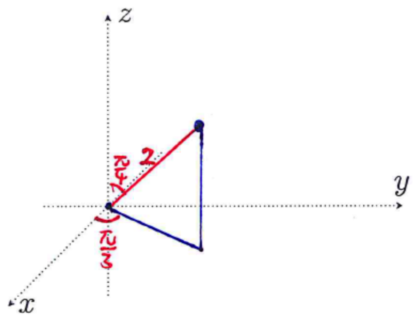
We can obtain formulas converting from rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2$$

Here ρ is the distance from origin to $P = (x, y, z)$.



Example 8. Find the rectangular coordinates of the point P with spherical coordinates $(2, \frac{\pi}{3}, \frac{\pi}{4})$.



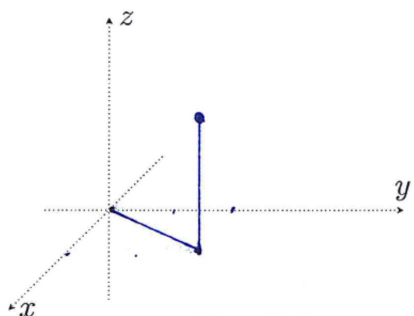
$$x=\rho\sin\phi\cos\theta = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$$

$$y=\rho\sin\phi\sin\theta = 2\sin\left(\frac{\pi}{4}\right)\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$$

$$z=\rho\cos\phi = 2\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$$

Example 9. Find spherical coordinates of the point with rectangular coordinates $(1, \sqrt{3}, 2)$.



$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+4} = 2\sqrt{2}$$

$$\cos\phi = \frac{z}{\rho} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

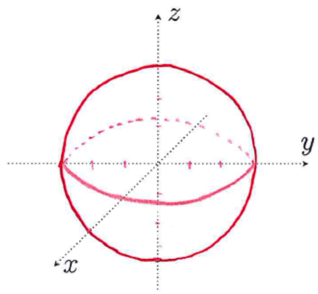
So $\phi = \frac{\pi}{4}$

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{1}{2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{2}$$

So $\theta = \frac{\pi}{3}$

$$\left(2\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$$

Example 10. Describe the surface with equation in spherical coordinates is $\rho = 3$.



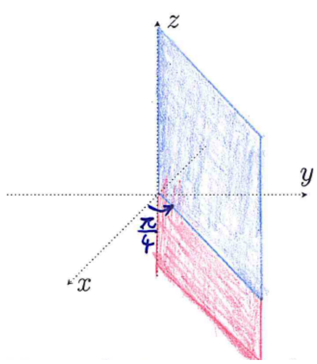
$\rho = 3$

$\sqrt{x^2+y^2+z^2} = 3$

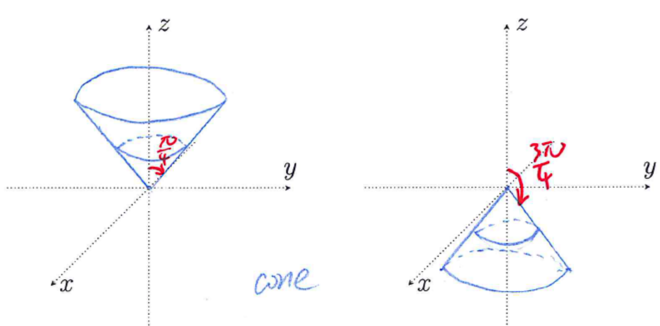
$x^2+y^2+z^2 = 9$

Sphere with radius 3

Example 11. Describe the surface with equation in spherical coordinates is $\theta = \pi/4$.



Example 12. Describe the surface with equation in spherical coordinates is (1) $\phi = \pi/4$. (2) $\phi = 3\pi/4$.

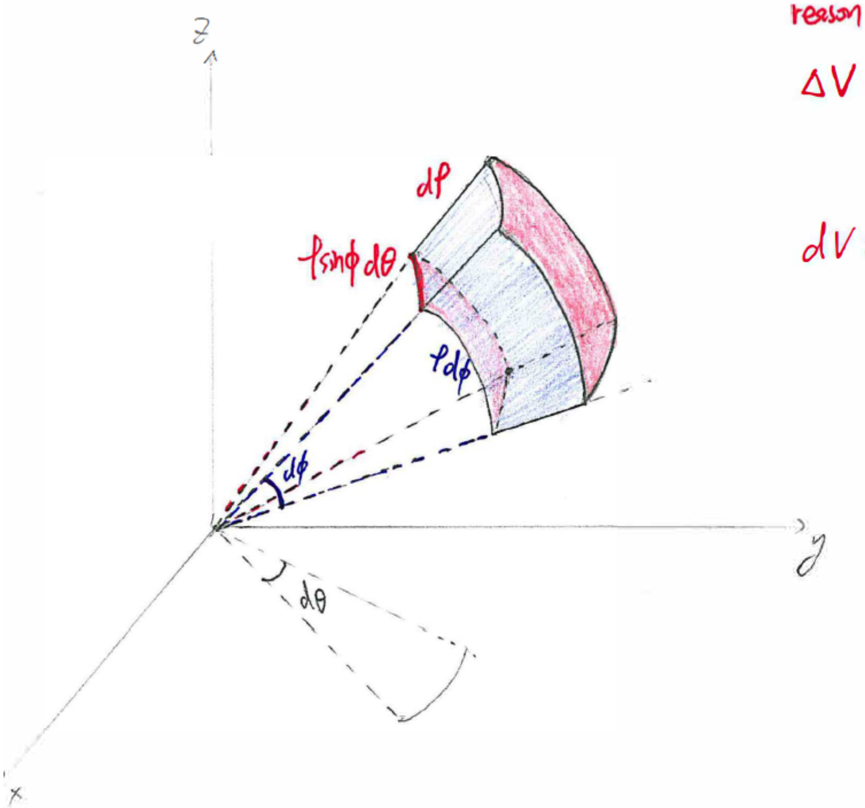


If a solid region E is defined by spherical coordinates:

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

The formula for triple integration in spherical coordinates:

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi.$$



reason:

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi\,\Delta\theta)$$
$$= \rho^2 \sin\phi\ \Delta\rho\,\Delta\theta\,\Delta\phi$$
$$dV = \rho^2 \sin\phi\ d\rho\, d\theta\, d\phi$$

Example 13. Evaluate $\iiint_B 3e^{(x^2+y^2+z^2)^{3/2}}\ dV$, where B is the ball

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$

$$= \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

$$\iiint_B 3e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 3e^{(\rho^2)^{\frac{3}{2}}} \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^{\pi} \sin\phi\, d\phi\right) \cdot \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_0^1 3e^{\rho^3}\rho^2\, d\rho\right)$$

$$= -\cos\phi \Big|_0^{\pi} \cdot (2\pi) \cdot e^{\rho^3}\Big|_0^1$$

$$= 2(2\pi)(e-1)$$

$$= 4\pi(e-1)$$

$u = \rho^3$

$du = 3\rho^2 d\rho$

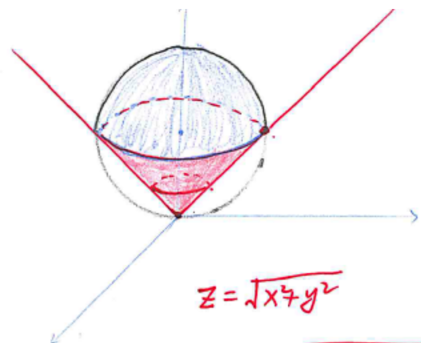
$$\int 3e^u \frac{du}{3}$$

$$= \int e^u du$$

$$= e^u + C$$

$$= e^{\rho^3} + C$$

Example 14. Evaluate $\iiint_E 1\, dV$, where E is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.



Spherical coordinates:

$$E = \left\{ (\rho, \theta, \phi) \middle| \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{4} \\ 0 \le \rho \le \cos\phi \end{array} \right\}$$

$z = \sqrt{x^2+y^2}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$= \sqrt{\rho^2\sin^2\phi}$

$= \rho\sin\phi$

$\Rightarrow \cos\phi = \sin\phi$

$\Rightarrow \phi = \frac{\pi}{4}$

$x^2+y^2+z^2 = z$

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$ or $\rho = 0$

$$V(E) = \iiint_E 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi \left. \frac{\rho^3}{3} \right|_0^{\cos\phi} d\phi$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\phi\cos^3\phi\, d\phi$$

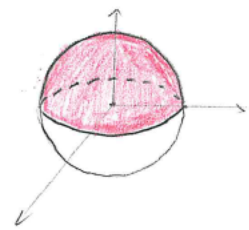
$$= \frac{2\pi}{3}\left. \left(-\frac{\cos^4\phi}{4} \right) \right|_0^{\frac{\pi}{4}}$$

$$= \frac{2\pi}{3}\left(-\frac{\left(\frac{\sqrt{2}}{2}\right)^4}{4} + \frac{1}{4} \right) = \frac{\pi}{8}$$

Example 15. Evaluate $\iiint_E (4 - x^2 - y^2)\, dV$, where B is the top half ball

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}$$

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}$$



$$B = \{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \tfrac{\pi}{2}\}$$

$$\iiint_E 4 - x^2 - y^2\, dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \left[4 - (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\right]\cdot \rho^2\sin\phi \; d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \left(4 - \rho^2\sin^2\phi\right)\rho^2\sin\phi\; d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^{\frac{\pi}{2}} \frac{4\rho^3}{3}\sin\phi - \frac{\rho^5}{5}\sin^3\phi\Big|_{\rho=0}^{\rho=1} d\phi$$

$$= 2\pi\cdot\int_0^{\frac{\pi}{2}} \frac{4}{3}\sin\phi - \frac{1}{5}\sin\phi(1-\cos^2\phi)\, d\phi$$

$$= 2\pi\left(-\frac{4}{3}\cos\phi + \frac{1}{5}\left(\cos\phi - \frac{\cos^3\phi}{3}\right)\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 2\pi\left(\frac{4}{3} - \frac{1}{5}\left(1-\frac{1}{3}\right)\right) = \frac{12\pi}{5}$$

Remark: (Not required)

To summarize, Polar coordinate, Cylindrical coordinate, Spherical coordinate are examples of change of coordinates.

More generally, if there is any (good) change of coordinate function $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$, then

$$\Phi = \langle x = \phi_1(r,s,t), y = \phi_2(r,s,t), z = \phi_3(r,s,t) \rangle$$

Define matrix

$$d\Phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial r} & \frac{\partial \phi_1}{\partial s} & \frac{\partial \phi_1}{\partial t} \\ \frac{\partial \phi_2}{\partial r} & \frac{\partial \phi_2}{\partial s} & \frac{\partial \phi_2}{\partial t} \\ \frac{\partial \phi_3}{\partial r} & \frac{\partial \phi_3}{\partial s} & \frac{\partial \phi_3}{\partial t} \end{bmatrix}$$

Then

$$dV = dxdydz = |\det(d\Phi)| drdsdt$$

So,

$$\boxed{\iiint\limits_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\phi_1(r,s,t), \phi_2(r,s,t), \phi_3(r,s,t))\ |\det(d\Phi)|\ dr\ ds\ dt.}$$