§3.4 Integration in $\mathbb{R}^3$ and $\mathbb{R}^n$

Motivation: density and mass...

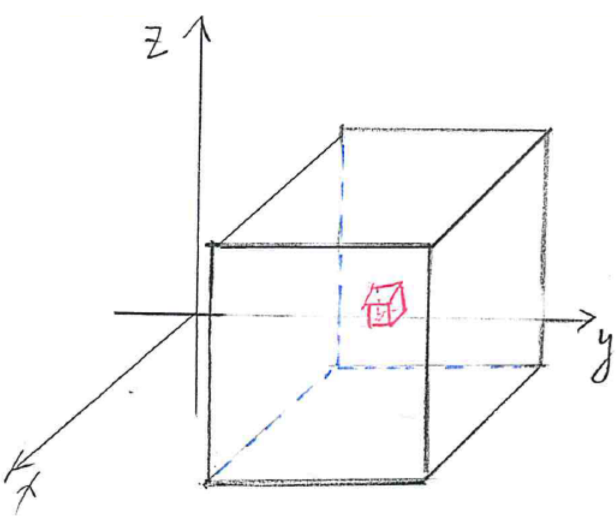
Suppose $f(x, y, z)$ is continuous on B, a bounded region in $\mathbb{R}^3$.

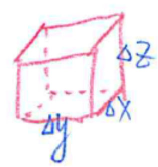
Definition.

The **triple integral** of $f(x, y, z)$ on $B \subset \mathbb{R}^3$ is defined as limits of Riemann sum:

$$\iiint_B f(x,y,z)\ dV = \lim_{\Delta V \to 0} \sum_{i,j,k} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V.$$

Here, $\Delta V = (\Delta x)(\Delta y)(\Delta z)$.





Theorem. Fubini's Theorem.

If $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z) dx dy dz.$$

$$\iiint_B f(x,y,z)\ dV = \int_r^s \int_a^b \int_c^d f(x,y,z) dy dx dz.$$

Example 1. Evaluate the triple integral

$$\iiint_B xy^2z \, dV,$$

where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 2, -2 \le y \le 1, 0 \le z \le 3\}.$$

$$\iiint_B xy^2z\,dV = \int_{-2}^{1}\int_0^2\int_0^3 xy^2z\,dz\,dx\,dy$$

$$= \int_{-2}^{1}\int_0^2 \frac{xy^2z^2}{2}\Big|_0^3\,dx\,dy$$

$$= \int_{-2}^{1}\int_0^2 \frac{9}{2}xy^2\,dx\,dy$$

$$= \int_{-2}^{1} \frac{9}{2}\cdot\frac{x^2}{2}y^2\Big|_0^2\,dy$$

$$= \int_{-2}^{1} 9y^2\,dy$$

$$= \frac{9y^3}{3}\Big|_{-2}^{1}$$

$$= 3 - 3(-2)^3$$

$$= 27$$

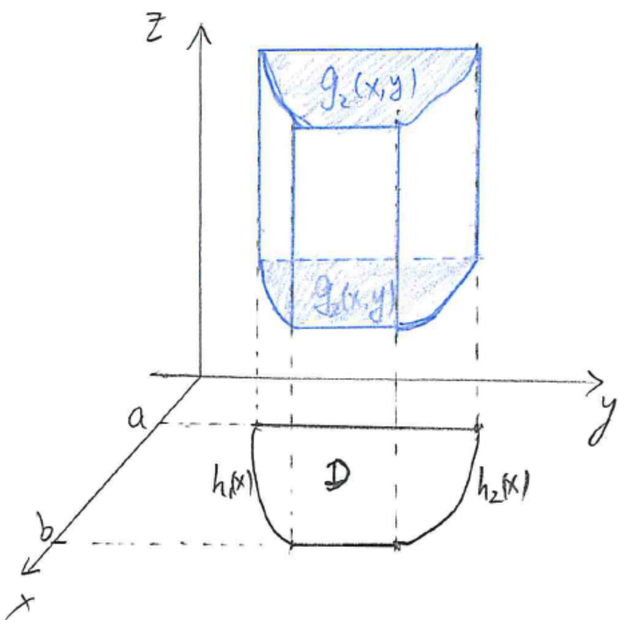
Theorem.

If a solid region E lies between the graphs of two functions:

$$E = \{(x, y, z) \mid (x, y) \in D, g_1(x, y) \leq z \leq g_2(x, y)\},$$

then,

$$\iiint_E f(x, y, z)\ dV = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) dz \right] dA.$$



In particular, if D is given by

$$D = \{(x, y) \mid a \leq x \leq b, h_1(x) \leq y \leq h_2(x)\},$$

then

$$\iiint_E f(x, y, z)\ dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) dz \right] dydx.$$

In particular, if the function $f(x, y, z)$ is the constant function 1, then the integration gives us the volume of the region E. (**§3.5 Volume** is only about this particular theorem)

Theorem. §3.5 Volume

$$\text{Volume of } E = \iiint_E 1\ dV$$

Example 2. Evaluate the triple integral

$$\iiint_E y\ dV,$$

where E is the solid region bounded by four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

$$z = 1 - x - y$$

$$\iiint_E y\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} yz \Big|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y - xy - y^2 \, dy\, dx$$

$$= \int_0^1 \frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3} \Big|_0^{1-x} dx$$

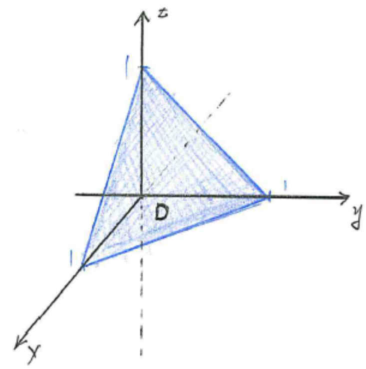
$$= \int_0^1 \frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3} \, dx$$

$$= \int_0^1 \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \, dx$$

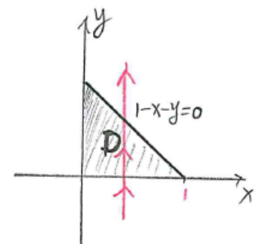
$$= \int_0^1 \frac{(1-x)^3}{6} \, dx$$

$$= -\frac{(1-x)^4}{24} \Big|_0^1$$

$$= 0 - \left(-\frac{1}{24}\right) = \frac{1}{24}$$



$$0 \le z \le 1 - x - y$$



$$D = \left\{ \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{array} \right\}$$

Example 3. (§3.5 Volume) Evaluate the triple integral

$$\iiint_E 1\, dV,$$

where E is the solid region bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

$$\iiint_E 1\, dV = \iint_D \left[\int_{x^2+y^2}^{8-x^2-y^2} 1\, dz \right] dA$$

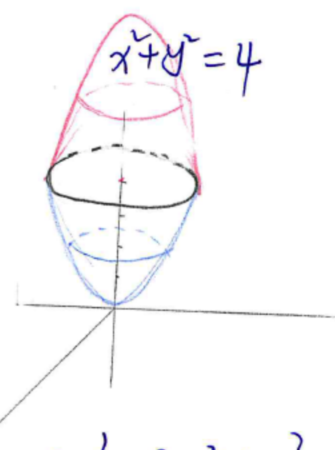
$$= \iint_D (8-x^2-y^2)-(x^2+y^2)\, dA$$

$$= \iint_D \left(8-2(x^2+y^2)\right) dA$$

Intersection: $\begin{cases} z = 8-x^2-y^2 \\ z = x^2+y^2. \end{cases}$

$\Rightarrow 8-x^2-y^2 = x^2+y^2$

$x^2+y^2=4$



Using polar coordinates:

$$= \int_0^{2\pi}\int_0^2 (8-2r^2)\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 8r-2r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} 4r^2 - \frac{r^4}{2}\Big|_0^2\, d\theta$$

$$= \int_0^{2\pi} 16-8\, d\theta = \int_0^{2\pi} 8\, d\theta = 8\theta\Big|_0^{2\pi}$$

$D: \{x^2+y^2 \le 4\}$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

This gives the volume of the solid region E.

$= 16\pi$

Question: Can you calculate the volume by double integral?

Theorem.

If a solid region E lies between the graphs of two functions:

$$E = \{(x, y, z) \mid (y, z) \in D, g_1(y, z) \le x \le g_2(y, z)\},$$

then,

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x, y, z) dx\right] dA.$$

Theorem.

If a solid region E lies between the graphs of two functions:

$$E = \{(x, y, z) \mid (x, z) \in D, g_1(x, z) \le y \le g_2(x, z)\},$$

then,

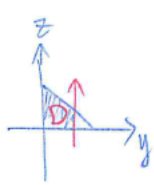
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x, y, z) dy\right] dA.$$

Example 4. Evaluate the triple integral

$$\iiint_E y\, dV,$$

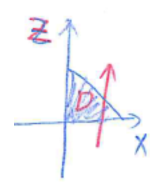
where E is the solid region bounded by four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Try the other two methods.



$$\iint_D \left[\int_0^{1-y-z} y\, dx\right] dA$$

$$= \int_0^1 \int_0^{1-y} \left[\int_0^{1-y-z} y\, dx\right] dz\, dy$$



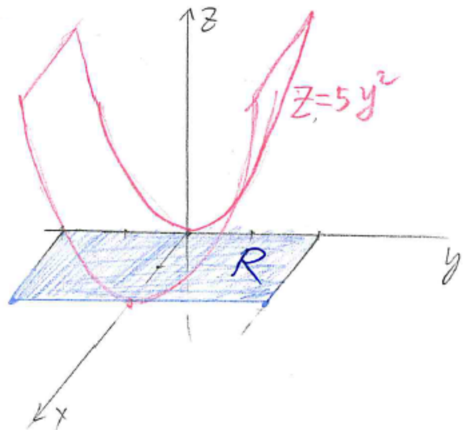
$$\iint_D \left[\int_0^{1-x-z} y\, dy\right] dA$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-z} y\, dy\right] dz\, dx$$

Example 5. (§3.5 Volume) Set up the integral to find the volume of the region between the cylinder $z = 5y^2$ and the xy-plane, which is also bounded by $x = 0$, $x = 2$, $y = -2$, $y = 2$.

(1) Use a double integral.

(2) Use a triple integral.



(1)

$$V = \iint_R 5y^2 \, dA$$

$$= \int_{-2}^{2} \int_{0}^{2} 5y^2 \, dx \, dy$$

(2) $$V = \iiint_E 1 \, dV = \iint_R \left[\int_0^{5y^2} 1 \, dz \right] dA$$

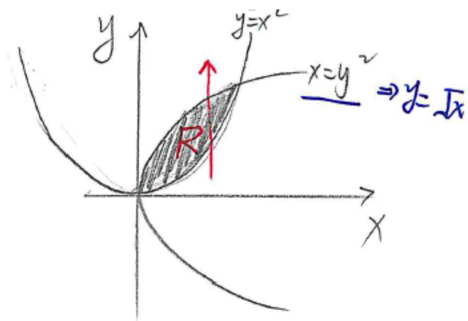
$$= \iint_R 5y^2 \, dA$$

Example 6. Evaluate the triple integral

$$\iiint_E x^2 \, dV,$$

where E is the solid region bounded by the parabolic cylinders $y = x^2$ and $x = y^2$, the plane $z = 2x + y$ and the xy-plane.

In the xy-plane, the parabolic cylinders intersect as shown in the picture. The solid E lies above the region R and below the plane $z = 2x + y$.



$$\iiint_E x^2 \, dV$$

$$= \iint_R \left[\int_0^{2x+y} x^2 \, dz \right] dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[\int_0^{2x+y} x^2 \, dz \right] dy \, dx$$

$$0 \leq x \leq 1$$

$$x^2 \leq y \leq \sqrt{x}$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 z \Big|_0^{2x+y} dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2x^3 + y x^2 \, dy \, dx$$

$$= \int_0^1 2x^3 y + \frac{y^2}{2} x^2 \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(2x^{7/2} + \frac{x^3}{2}\right) - \left(2x^5 + \frac{x^6}{2}\right) dx$$

$$= \frac{2x^{9/2}}{9/2} + \frac{x^4}{2(4)} - \frac{2x^6}{6} - \frac{x^7}{2(7)} \Big|_0^1$$

$$= \frac{4}{9} + \frac{1}{8} - \frac{1}{3} - \frac{1}{14} \quad (\text{ok})$$

$$= \frac{83}{504}$$

Example 7. Set up the integral to find the volume of the region in the first octant bounded by the coordinates planes, the plane $y + z = 7$ and the cylinder $x = 49 - y^2$.

$$\iiint_E 1\, dV$$

$$= \iint_R \left[\int_0^{7-y} 1\, dz \right] dA$$

$$= \iint_R 7-y\ dA$$

$$= \int_0^7 \int_0^{49-y^2} 7-y\ dx\, dy$$

$$= \int_0^7 (7-y)\, x \Big|_0^{49-y^2} dy$$

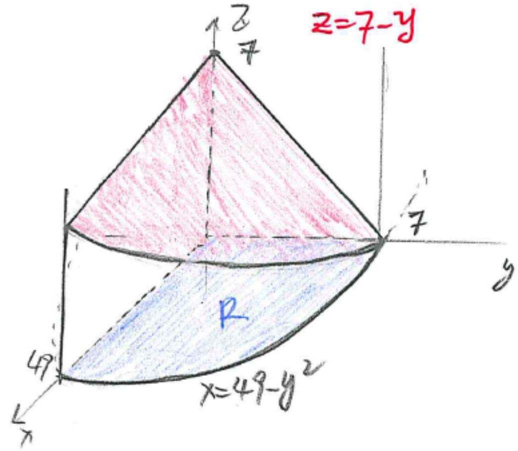
$$= \int_0^7 (7-y)(49-y^2)\, dy$$

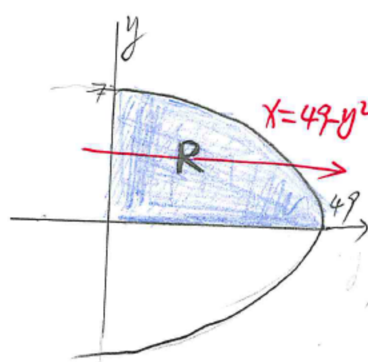
$$= \int_0^7 343 - 7y^2 - 49y + y^3\, dy$$

$$= 343y - \frac{7y^3}{3} - \frac{49y^2}{2} + \frac{y^4}{4} \Big|_0^7$$

$$= 7^4 - \frac{7^4}{3} - \frac{7^4}{2} + \frac{7^4}{4}$$

$$= \frac{5(7^4)}{12}$$





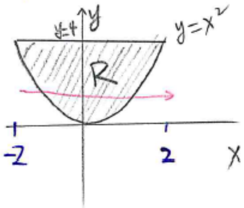
Example 8. Let E be the region bounded by $y = x^2$, $z = 0$, and $y + z = 4$.

Express the integral

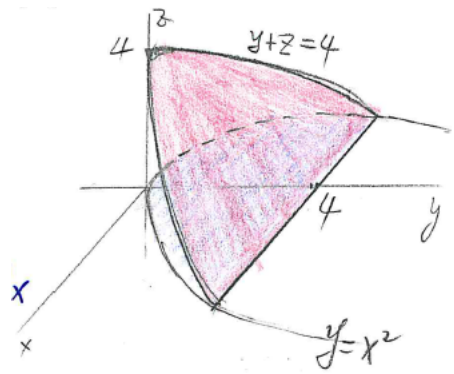
$$\iiint_E f(x, y, z)\, dV,$$

as an iterated integral in all 6 different ways.

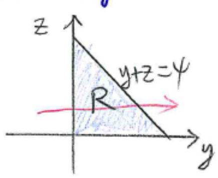
(I) $\cdot \displaystyle\iint_R \left[\int_0^{4-y} f\,dz\right] dA = \int_{-2}^{2}\int_{x^2}^{4}\left[\int_0^{4-y} f\,dz\right] dy\,dx$



$$= \int_0^4\int_{-\sqrt{x}}^{\sqrt{x}}\left[\int_0^{4-y} f\,dz\right] dx\,dy$$

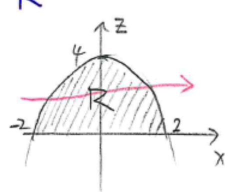


(II) $\cdot \displaystyle\iint_R \left[\int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\right] dA = \int_0^4\int_0^{4-y}\left[\int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\right] dz\,dy$



$$= \int_0^4\int_0^{4-z}\left[\int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\right] dy\,dz$$

(III) $\cdot \displaystyle\iint_R \left[\int_{x^2}^{4-z} f\,dy\right] dA = \int_{-2}^{2}\int_0^{4-x^2}\left[\int_{x^2}^{4-z} f\,dy\right] dz\,dx$



$\begin{cases} y = x^2 \\ y+z=4 \end{cases} \Rightarrow z = 4 - x^2$

$$= \int_0^4\int_{-\sqrt{4-z}}^{\sqrt{4-z}}\left[\int_{x^2}^{4-y} f\,dy\right] dx\,dz$$