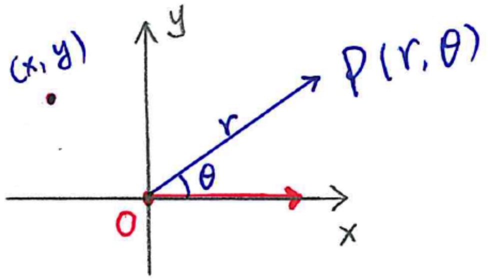
§3.3 Polar coordinates

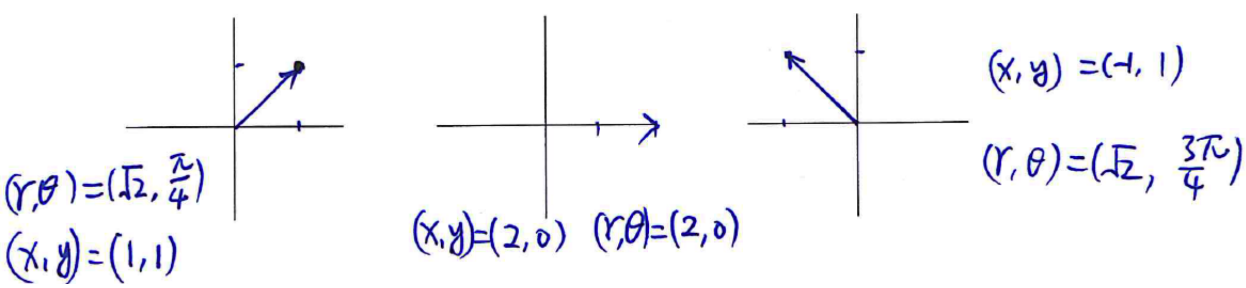
1. Polar Coordinates System for $\mathbb{R}^2$

For each point in $\mathbb{R}^2$, usually we use xy-Cartesian coordinates. However, sometimes, it is more convenient to use Polar Coordinates System.



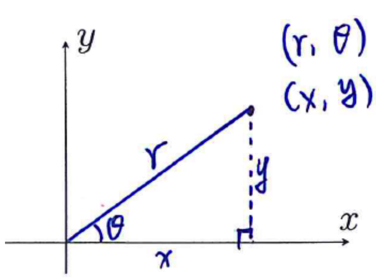
Each point P is describe by an ordered pare (r, θ), where r is distance from the origin O to P, and θ is the angle (usually measured in radians) between the polar axis and OP.

Examples



Using trigonometric formulas, relation between the (x, y)-Cartesian coordinates, and the (r, θ)-Polar Coordinates are related by

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$



Example 1.

(1) Equation $x^2 + y^2 = 25$ is equivalent to $r = 5$

(2) Equation $(x-2)^2 + y^2 = 4$ is equivalent to

$x^2 - 4x + 4 + y^2 = 4$

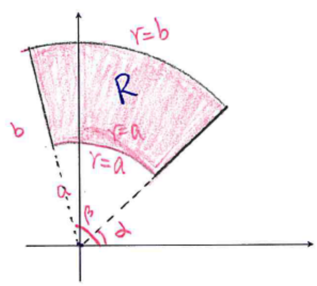
$\underline{x^2+y^2} - 4x = 0$ $\qquad$ $r = 0$ or $r = 4\cos\theta$

$r^2 - 4r\cos\theta = 0$

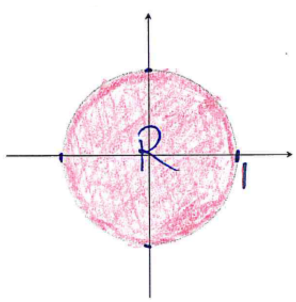
$r(r - 4\cos\theta) = 0$

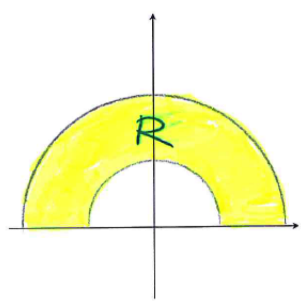
The region of a **polar rectangle**

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$



Example 2. $R = \{(x, y) | x^2 + y^2 \le 1\}$ and $R = \{(x, y) | 1 \le x^2 + y^2 \le 4, y \ge 0\}$

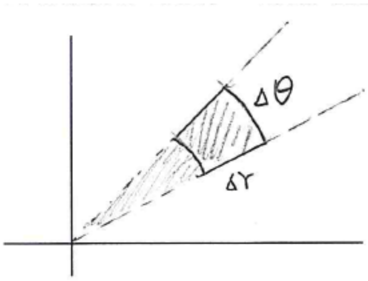


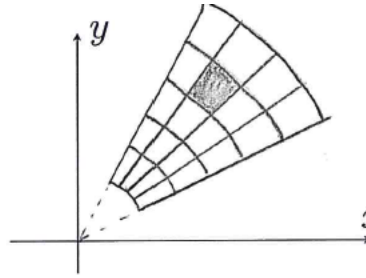


$R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ $\qquad$ $R = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$R = \{(x, y) \mid x^2 + y^2 \le 1\}$ $\qquad$ $R = \left\{(x, y) \mid \begin{matrix} x^2+y^2 \le 4 \\ x^2+y^2 \ge 1 \end{matrix} \quad y \ge 0\right\}$

2. Double integrals in polar form.





$$\pi r^2 \cdot \left(\frac{\Delta\theta}{2\pi}\right) = \text{Area} = \frac{1}{2} r^2 \Delta\theta$$

$$dA = r\,dr\,d\theta$$

Change to Polar Coordinates in Double Integral:

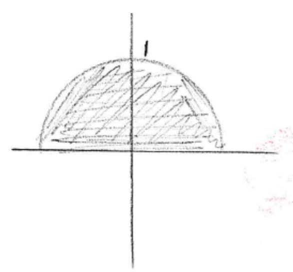
Theorem.

If the polar region R is given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$ for $0 \le \beta - \alpha \le 2\pi$, then

$$\iint\limits_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ drd\theta$$

Example 3. Change the integral $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 4\ dydx$ into polar integral.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 4\,dy\,dx = \int_0^{\pi}\int_0^1 4\,r\,dr\,d\theta$$



$$= \int_0^{\pi} 2r^2\Big|_0^1\,d\theta$$

$$\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi\}$$

$$= \int_0^{\pi} 2\,d\theta = 2\theta\Big|_0^{\pi} = 2\pi$$

Example 4. Express the double integral $\iint_R f(x,y)\,dydx$ into polar integral. Here R is given by $R = \{(x,y) \mid (x-2)^2 + y^2 \le 4\}$.

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

$$= \int_0^{\frac{1}{2}\pi} \int_0^{4\cos\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

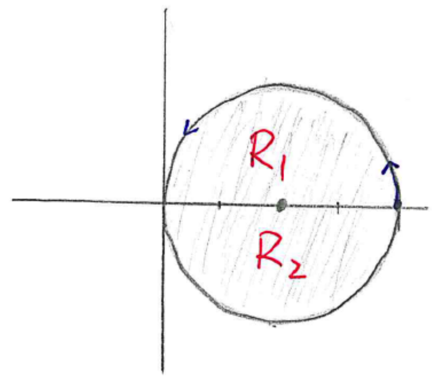
$$+ \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{4\cos\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$x^2 + y^2 - 4x \le 0$

$r^2 - 4r\cos\theta \le 0$

$r(r - 4\cos\theta) \le 0$

$0 \le r \le 4\cos\theta$



$0 \le \theta \le \frac{\pi}{2}$

and $\frac{3\pi}{2} \le \theta \le 2\pi$

Example 5. Evaluate $\iint_R \cos(x^2+y^2)dA$ where R is the region between $r=3$ and $r=4$ in the upper half plane $y \geq 0$.

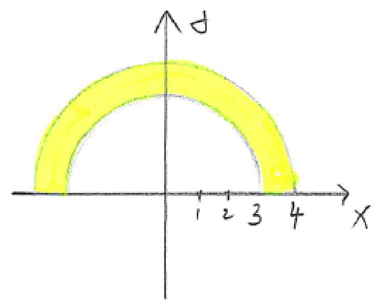
$$\iint_R \cos(x^2+y^2)\,dA$$

$$= \int_0^{\pi}\int_3^4 (\cos r^2)\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(\sin 16 - \sin 9)\,d\theta$$

$$= \frac{1}{2}(\sin 16 - \sin 9)\,\theta\Big|_0^{\pi}$$

$$= \frac{1}{2}(\sin 16 - \sin 9)\pi$$



$3 \leq r \leq 4$

$0 \leq \theta \leq \pi$

$$\int_3^4 (\cos r^2) r\,dr$$

$$= \frac{1}{2}\sin r^2\Big|_3^4$$

$$= \frac{1}{2}(\sin 16 - \sin 9)$$

$u = r^2$

$du = 2r\,dr$

$$\int (\cos r^2) r\,dr$$

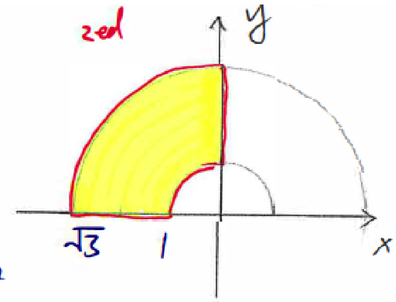
$$= \int (\cos u)\frac{1}{2}du$$

$$= \frac{1}{2}\sin u$$

$$= \frac{1}{2}\sin r^2$$

Example 6. Evaluate $\iint_R 4x^2 + 3y dA$ where R is the region in the 2ed quadrant bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 3$.

$$\iint_R 4x^2+3y \, dA$$



$$= \int_{\frac{\pi}{2}}^{\pi} \int_1^{\sqrt{3}} (4r^2\cos^2\theta + 3r\sin\theta) r \, dr \, d\theta$$

$1 \leq r \leq \sqrt{3}$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_1^{\sqrt{3}} 4r^3\cos^2\theta + 3r^2\sin\theta \, dr \, d\theta$$

$\frac{\pi}{2} \leq \theta \leq \pi$

$$= \int_{\frac{\pi}{2}}^{\pi} r^4\cos^2\theta + r^3\sin\theta \Big|_1^{\sqrt{3}} \, d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} 8\cos^2\theta + (3\sqrt{3}-1)\sin\theta \, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= \int_{\frac{\pi}{2}}^{\pi} 4(\cos 2\theta + 1) + (3\sqrt{3}-1)\sin\theta \, d\theta$$

$$= 2\sin 2\theta + 4\theta - (3\sqrt{3}-1)\cos\theta \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\pi + (3\sqrt{3}-1) - 2\pi$$

$$= 2\pi + (3\sqrt{3}-1)$$

Example 7. Use polar coordinates to find the volume of the given solid below the paraboloid $z = 6 - 2x^2 - 2y^2$ and above the first quadrant of the xy-plane.

To find the region R, set $z=0$.

$$V = \iint_R 6-2x^2-2y^2 \, dA$$

$$= 2\iint_R 3-(x^2+y^2) \, dA$$

$$= 2\int_0^{\pi/2}\int_0^{\sqrt{3}} (3-r^2) \, r\,dr\,d\theta$$

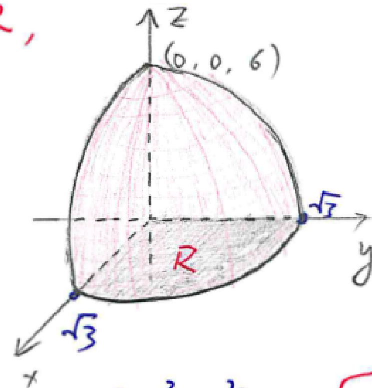
$$= 2\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{3}} 3r-r^3 \, dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left.\frac{3r^2}{2}-\frac{r^4}{4}\right|_0^{\sqrt{3}} d\theta$$

$$= 2\int_0^{\pi/2} \frac{9}{2}-\frac{9}{4} \, d\theta$$

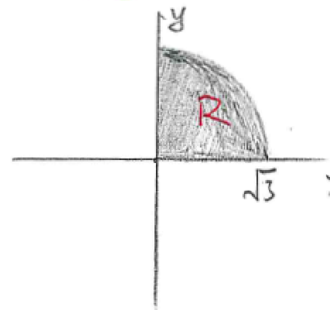
$$= 2\int_0^{\pi/2} \frac{9}{4} \, d\theta$$

$$= 2\left(\frac{9}{4}\theta\right)\Big|_0^{\pi/2} = \frac{9}{2}\cdot\frac{\pi}{2} = \frac{9\pi}{4}$$



$6-2x^2-2y^2=0$ $z=0$

$x^2+y^2=3$



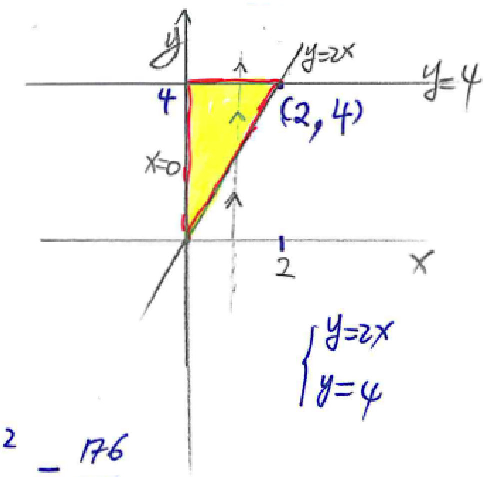
Example 8. Find the volume of the given solid. Enclosed by the paraboloid $z = 2x^2 + y^2$ and the planes $x = 0, y = 4, y = 2x, z = 0$

$$V = \iint_R 2x^2 + y^2 \, dA$$

$$= \int_0^2 \int_{2x}^4 2x^2 + y^2 \, dy \, dx$$

$$= \int_0^2 2x^2 y + \frac{y^3}{3} \Big|_{2x}^4 \, dx$$

$$= \int_0^2 8x^2 + \frac{64}{3} - \frac{4}{3}x^3 \, dx = \frac{8x^3}{3} + \frac{64}{3}x - \frac{x^4}{3} \Big|_0^2 = \frac{176}{3}$$



$\begin{cases} y = 2x \\ y = 4 \end{cases}$

Example 9. Use polar coordinates to find the volume of the given solid. Inside the sphere $x^2 + y^2 + z^2 = 25$ and outside the cylinder $x^2 + y^2 = 9$

We calculate the top half volume then times 2. $(V = 2V_{top})$

$$V_{top} = \iint_R \sqrt{25 - (x^2 + y^2)} \, dA$$

$$= \int_0^{2\pi} \int_3^5 \sqrt{25 - r^2} \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_3^5 (25 - r^2)^{\frac{1}{2}} \frac{1}{2} dr^2 \, d\theta$$

$$= \int_0^{2\pi} - \frac{(25 - r^2)^{\frac{3}{2}}}{3/2} \cdot \frac{1}{2} \Big|_3^5 \, d\theta$$

$$= \int_0^{2\pi} \frac{64}{3} \, d\theta = \frac{64}{3} \cdot 2\pi$$