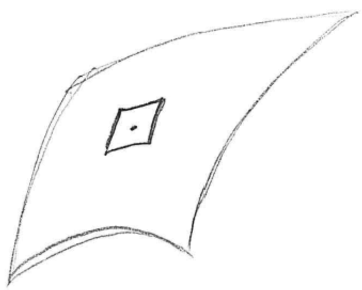
§3.11 Surface and Area

Let S be a surface with a vector equation $\vec{r} = \vec{r}(u, v) = \langle r_1, r_2, r_3 \rangle$ from $\mathbb{R}^2$ to $\mathbb{R}^3$.

The area of S on domain D is
$$A(S) = \lim_{\Delta A \to 0} \sum_{i,j} \Delta T_{ij}$$
where ΔT_{ij} is the area of the tangent plane on ΔA_{ij}.



At each point on the surface, there are two tangent vectors $\vec{r}_u$ and $\vec{r}_v$. The area is
$$\Delta T_{ij} = |\vec{r}_u du \times \vec{r}_v dv| = |\vec{r}_u \times \vec{r}_v| du dv$$

Theorem.

The **area of the surface** S can be computed by
$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| du dv.$$

Example 1. Find the area of the surface S with parametric equation $\vec{r} = \langle u \cos v, u \sin v, u^2 \sin(2v) \rangle$ where $0 \leq u \leq \sqrt{2}$ and $0 \leq v \leq 2\pi$.

$\vec{r}_u = \langle \cos v, \sin v, 2u \sin(2v) \rangle$ and $\vec{r}_v = \langle -u \sin v, u \cos v, 2u^2 \cos(2v) \rangle$
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \sin(2v) \\ -u \sin v & u \cos v & 2u^2 \cos(2v) \end{vmatrix} = \langle 2u^2(-\sin v), -2u^2 \cos v, u \rangle$$
So, $|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}$.
Hence
$$A(S) = \int_0^{2\pi} \int_0^{\sqrt{2}} u\sqrt{4u^2 + 1} du dv = \ldots = \frac{32\pi}{3}$$

Let S be a surface with equation $z = f(x, y)$.

Theorem.

The **area of the surface** S can be computed by

$$A(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

Reason: A parametrization of the surface is $\vec{r}(x, y) = \langle x, y, f(x, y)\rangle$. So, $\vec{r}_x = \langle 1, 0, f_x\rangle$ and $\vec{r}_y = \langle 0, 1, f_y\rangle$. Hence computations the cross product $\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1\rangle$. We relate ΔT_{ij} and ΔA_{ij} by

$$\Delta T_{ij} = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\; \Delta A$$

Example 2. Find the area of the plane $z = 8+2x+5y$ that lies above the rectangle $[0, 3]\times[1, 6]$.

Denote $f(x, y) = 8 + 2x + 5y$. Then $f_x = 2$ and $f_y = 5$.

$$A(S) = \int_0^3 \int_1^6 \sqrt{4+25+1}\, dy\, dx$$

$$= \int_0^3 \int_1^6 \sqrt{30}\, dy\, dx$$

$$= \int_0^3 \sqrt{30}y\Big|_1^6\, dx$$

$$= \int_0^3 5\sqrt{30}\, dx$$

$$= 5\sqrt{30}x\Big|_0^3 = 15\sqrt{30}$$

Example 3. Find the area of the surface. The part of the surface $z = 2xy$ that lies within the cylinder $x^2 + y^2 = 2$

$f_x = 2y$ $\quad$ $0 \leq r \leq \sqrt{2}$

$f_y = 2x$ $\quad$ $0 \leq \theta \leq 2\pi$

$$A(S) = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4r^2+1)^{\frac{1}{2}}\, \underline{r}\, \underline{dr}\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4r^2+1)^{\frac{1}{2}}\, \frac{1}{8} d(4r^2+1)\, d\theta$$

$$= \int_0^{2\pi} d\theta \left(\frac{(4r^2+1)^{\frac{3}{2}}}{\frac{3}{2}} \right) \Bigg|_0^{\sqrt{2}}$$

$$= 2\pi \left(6 - \frac{2}{3} \right)$$

$$= \frac{32\pi}{3}$$

Example 4. Find the area of the surface. The part of the plane $2x + 3y + z = 8$ that lies in the first octant.

$z = 8 - 2x - 3y$

$f_x = -2$

$f_y = -3$

$$A(S) = \iint_D \sqrt{4+9+1}\, dA$$

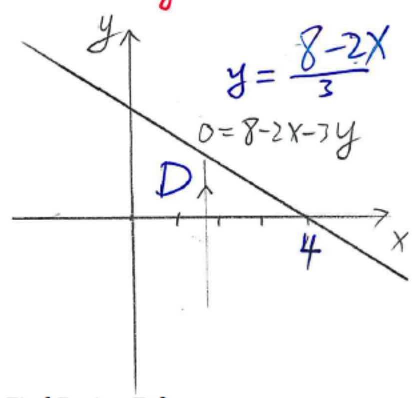
$$= \int_0^4 \int_0^{\frac{8-2x}{3}} \sqrt{14}\, dy\, dx$$

$$= \int_0^4 \sqrt{14}\, \frac{8-2x}{3}\, dx$$

$$= \frac{8\sqrt{14}}{3} x - \frac{\sqrt{14}}{3} x^2 \Big|_0^4$$

$$= \frac{32\sqrt{14}}{3} - \frac{16\sqrt{14}}{3}$$

$$= \frac{16}{3}\sqrt{14}$$



Find Region D first.
To find region D, set z=0.

Example 5. Find the surface area of a sphere of radius r

Spherical system: $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$$D = \begin{Bmatrix} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{Bmatrix}$$

parametric domain.

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle r^2\sin^2\phi\cos\theta,\ r^2\sin^2\phi\sin\theta,\ r^2\sin\phi\cos\phi \right\rangle$$

$$\left|\vec{r}_\phi \times \vec{r}_\theta\right| = \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi} = r^2\sqrt{\sin^2\phi} = r^2\sin\phi$$

$(\sin\phi \ge 0 \text{ since } 0 \le \phi \le \pi)$

$$\text{Area} = \iint_D |\vec{r}_\phi \times \vec{r}_\theta|\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\, d\phi\, d\theta = r^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\, d\phi$$

$$= 2\pi r^2(-\cos\phi)\Big|_0^{\pi}$$

$$= 4\pi r^2$$