§3.1-3.2 Double integral and Iterated integral

• **Review**: Single integral over interval.

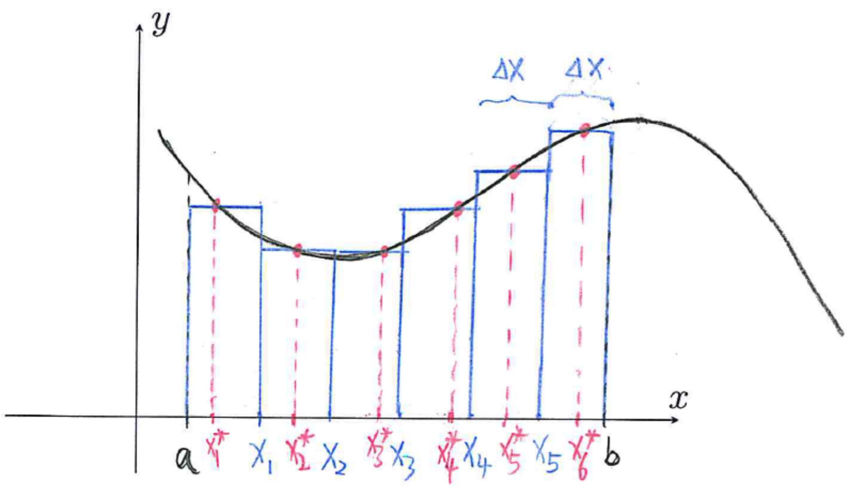
1. Definition.

Definition.

The **definite integral** of a continuous function $f(x)$ on $[a, b]$ is defined as the limit of Riemann sum:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_i f(x_i^*)\Delta x.$$

2. Estimation. Definite integral of $f(x)$ from a to b can be estimated by Riemann sum $\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$



We may use left, right, or mid-points estimations.

3. Calculation.

Theorem. The Fundamental Theorem of Calculus.

If $F(x)$ is any anti-derivative of a continuous function $f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1. Find $\int_1^7 x^2 dx$.

Solution: $\int_1^7 x^2 dx = \left[\frac{x^3}{3}\right]_1^7 = 114.$

• Double integral of $f(x,y)$ on region R

1. Definition. Riemann integral (concepts)

Definition.

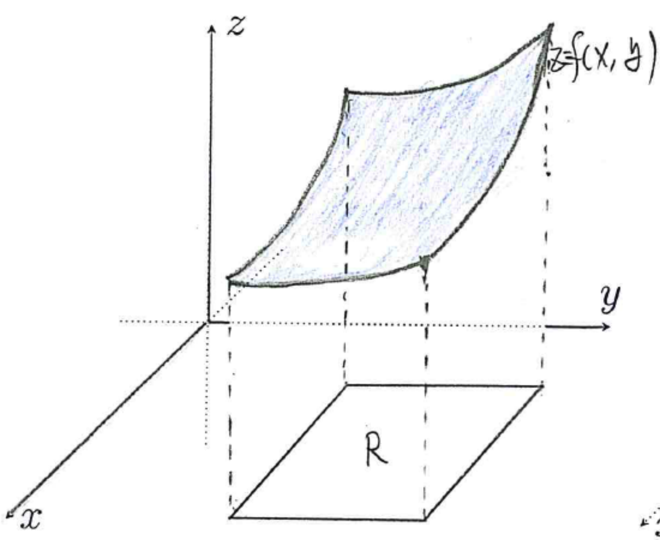
The **double integral** of $f(x,y)$ on R is defined as limits of Riemann sum:

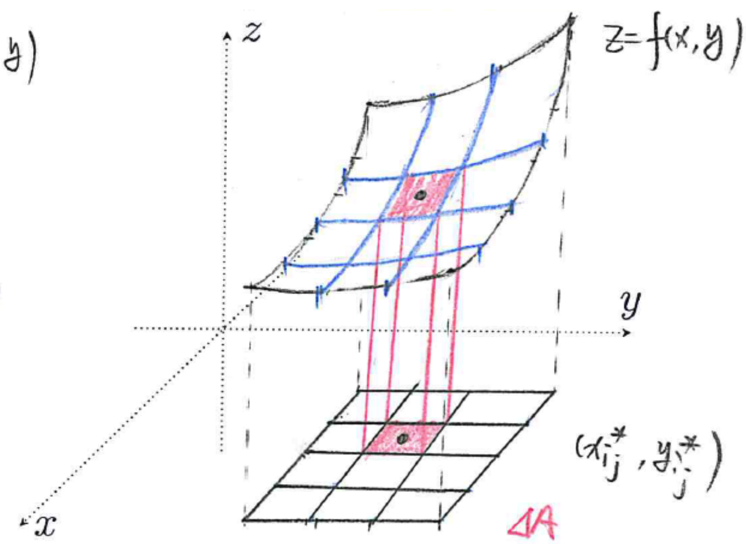
$$\iint_R f(x,y)\, dA = \lim_{\Delta A \to 0} \sum_{i,j} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Here, $\Delta A = (\Delta x)(\Delta y)$.

Geometric meaning:

If $f(x,y)$ is positive, then $\iint_R f(x,y)\, dA$ is the volume of the columns cylinder between $f(x,y)$ and xy-plane on area R.



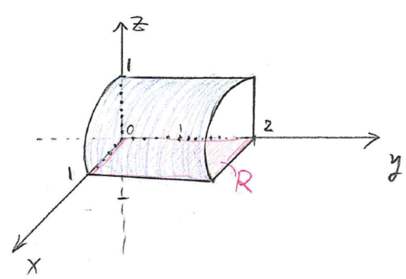


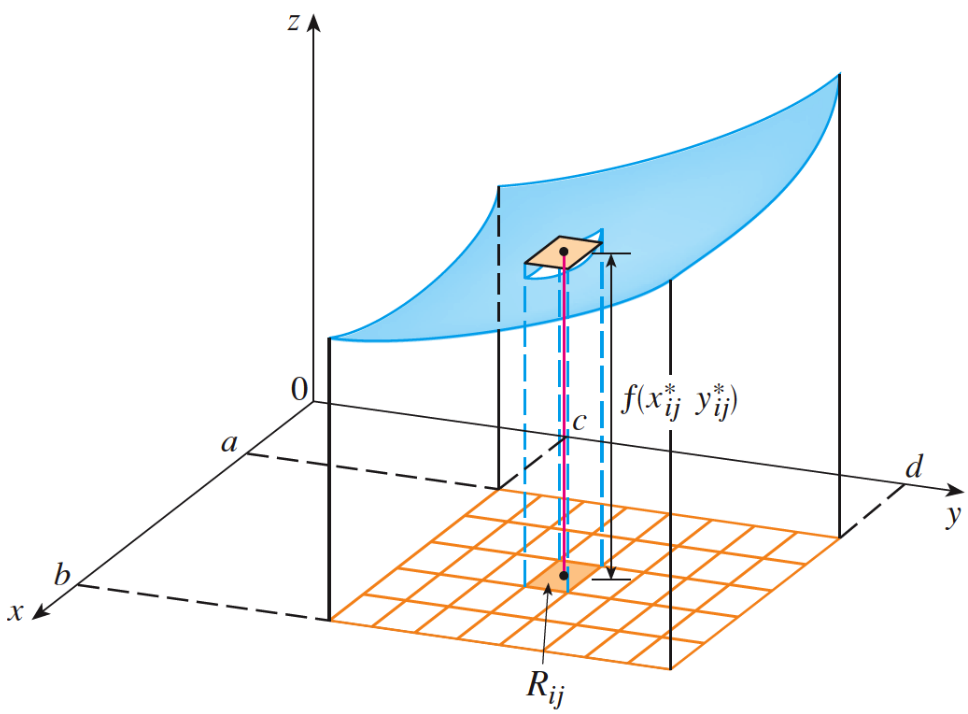
Example 2. Evaluate the double integral $\iint_R \sqrt{1-x^2}\, dA$ by first identifying it as the volume of a solid.

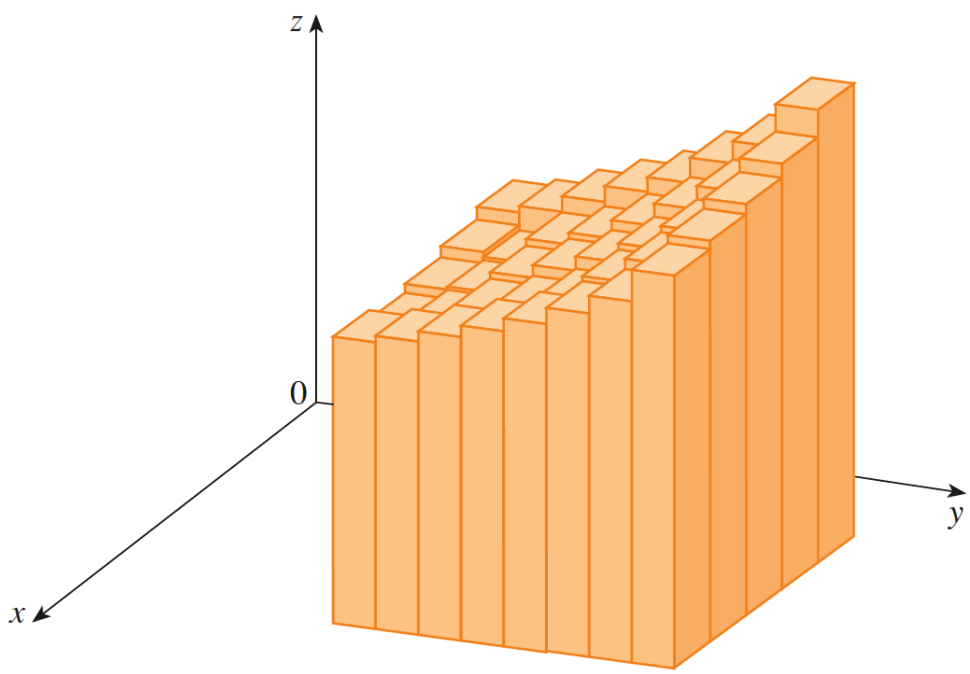
$z = f(x,y) = \sqrt{1-x^2}$ implies $x^2 + z^2 = 1$ which is a cylinder.

Volume of the cylinder on R is $V = \frac{1}{4}(\pi r^2)h = \frac{\pi}{2}$.

So, $\iint_R \sqrt{1-x^2}\, dA = \frac{\pi}{2}$




z
0
a
b
x
c
d
y
$f(x_{ij}^*, y_{ij}^*)$
R_{ij}


z
0
x
y

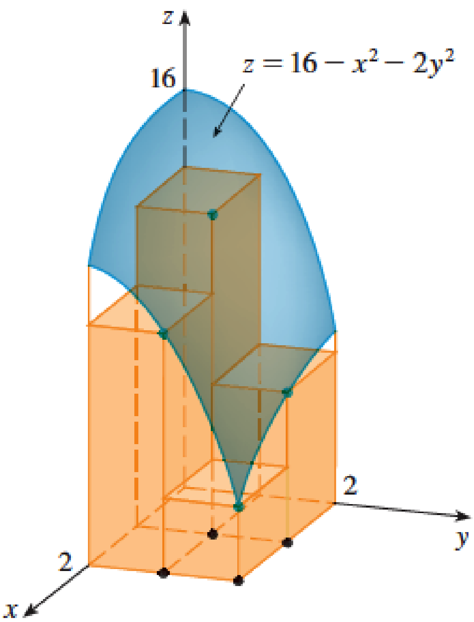
2. Estimations:

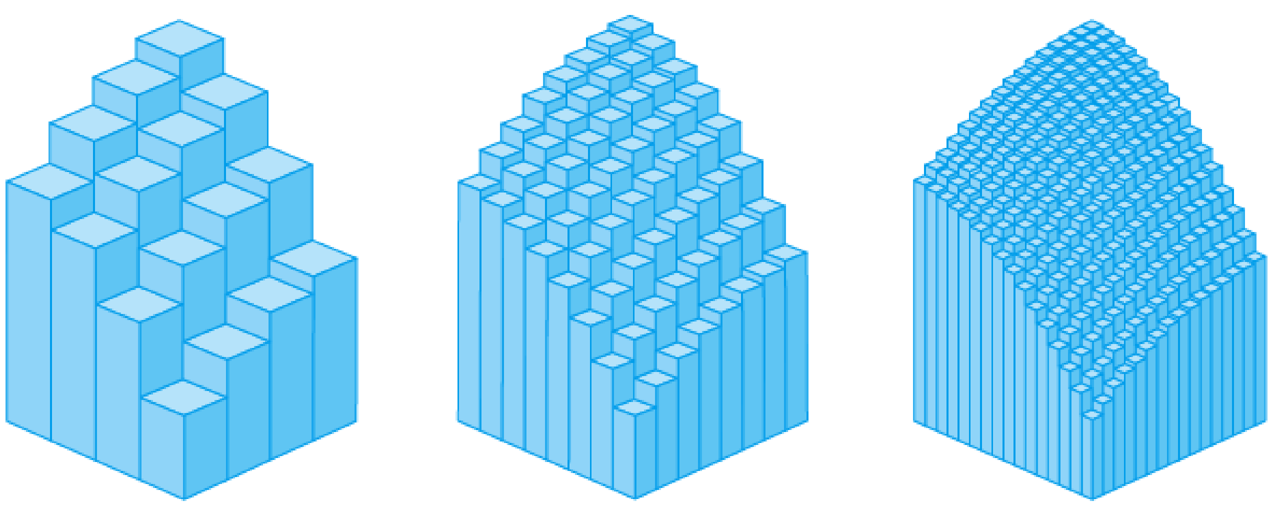
Example 3. Estimate the **volume** of the solid that lies below the surface

$$z = 16 - x^2 - 2y^2$$

and above the following rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}$

$$Volume = \iint_R f(x, y)\, dA \approx \sum_{i,j} f(x_{ij}^*, y_{ij}^*)\Delta A.$$





(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Example 4. Estimate the **volume** of the solid that lies below the surface $z = x^2y$ and above the following rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\}$

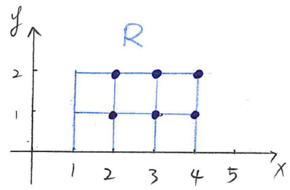
Use a Riemann sum with $m = 3, n = 2$, and take the sample point to be the upper right corner of each square.

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= f(2,1)\Delta A + f(2,2)\Delta A + f(3,1)\Delta A + f(3,2)\Delta A + f(4,1)\Delta A + f(4,2)\Delta A$$

$$= 4\times 1 + 8\times 1 + 9\times 1 + 18\times 1 + 16\times 1 + 32\times 1$$

$$= 87$$



$\Delta x = \frac{4-1}{m} = \frac{3}{3} = 1$

$\Delta y = \frac{2-0}{n} = \frac{2}{2} = 1$

$\Delta A = \Delta x \Delta y = 1$

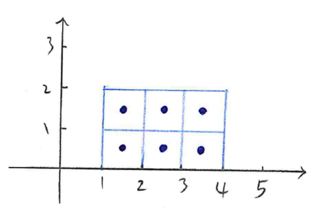
The Midpoint Rule:

The double integral of $f(x, y)$ on R can be estimated as

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i) \Delta A.$$

Here, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_i$ is the midpoint of $[y_{i-1}, y_i]$.

Example 5. Use a Midpoint Rule with $m = 3, n = 2$. Estimate the double integral of $f(x, y) = x^2y$ on the rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\}$



$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j) \Delta A$$

$$= f(1.5, 0.5)\Delta A + f(1.5, 1.5)\Delta A + f(2.5, 0.5)\Delta A + f(2.5, 1.5)\Delta A + f(3.5, 0.5)\Delta A + f(3.5, 1.5)\Delta A$$

$$= 41.5$$

$\Delta x = 1$

$\Delta y = 1$

$\Delta A = \Delta x \Delta y = 1$

3. Calculations: (Fubini's Theorem)

Suppose $f = f(x,y)$ is continuous on the rectangle $R = [a,b] \times [c,d]$.

Iterated integrals.

• Similarly as partial derivative, we can calculate **Partial Integral** $\int_a^b f(x,y)\,dx$ respect to x by thinking y as constant.

Example 6. $f(x,y) = x^2y$ for $1 \le x \le 4$ and $0 \le y \le 2$. Find $\int_a^b f(x,y)\,dx$

$$\int_a^b f(x,y)\,dx = \int_1^4 x^2y dx = \left[\frac{x^3}{3}y\right]_{x=1}^{x=4} = \frac{64}{3}y - \frac{1}{3}y = 21y.$$

Definition.

The **iterated integral** is

$$\int_c^d \int_a^b f(x,y)\,dxdy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Example 7. $f(x,y) = x^2y$ for $1 \le x \le 4$ and $0 \le y \le 2$. Find $\int_c^d \int_a^b f(x,y)\,dxdy$.

$$\int_c^d \int_a^b f(x,y)\,dxdy = \int_0^2 (\int_1^4 x^2y dx)dy = \int_0^2 21y dy = \left[21\frac{y^2}{2}\right]_0^2 = 42$$

Theorem. Fubini's Theorem.

The double integral can be calculated by iterated integrals:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$$

Example 8. $f(x,y) = x^2y$ for $1 \le x \le 4$ and $0 \le y \le 2$. Find $\iint_R f(x,y)\,dA$.

$$\iint_R f(x,y)\,dA = \int_0^2 \int_1^4 x^2y dxdy = 42$$

Example 9. Calculate the double integral of $f(x, y) = x^2y$ on the region $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x \le 4 \text{ and } 0 \le y \le 2\}$

$$\iint_R f(x,y)\,dA = \int_1^4 \left(\int_0^2 x^2 y\,dy\right) dx$$

$$= \int_1^4 \left(\frac{x^2y^2}{2}\Big|_0^2 \right) dx$$

$$= \int_1^4 2x^2\,dx = \frac{2x^3}{3}\Big|_1^4 = 42$$

Example 10. The region R is given by $0 \le x \le 2$ and $1 \le y \le 3$.

Calculate the double integral $\iint_R f(x, y)\,dA$ for $f(x, y) = 1 - x^2 - y^2$.

$$\iint_R f(x,y)\,dA = \int_1^3 \int_0^2 1 - x^2 - y^2\,dx\,dy$$

$$= \int_1^3 \left(\left(x - \frac{x^3}{3} - xy^2\right)\Big|_0^2 \right) dy$$

$$= \int_1^3 -\frac{2}{3} - 2y^2\,dy$$

$$= -\frac{2}{3}y - \frac{2y^3}{3}\Big|_1^3 = -\frac{56}{3}$$

Example 11. Calculate the iterated integral $\int_{-1}^{1} \int_0^{\pi/2} f(x, y)\,dxdy$ for $f(x, y) = 2y + y^3 \cos x$.

$$\int_0^{\pi/2} 2y + y^3\cos x \, dx = 2yx + y^3\sin x \Big|_0^{\pi/2} = \pi y + y^3$$

$$\int_{-1}^{1}\int_0^{\pi/2} f(x,y)\,dx\,dy = \int_{-1}^{1} \pi y + y^3 \, dy$$

$$= \frac{\pi y^2}{2} + \frac{y^4}{4}\Big|_{-1}^{1}$$

$$= 0$$

Example 12. Calculate the iterated integral

$$\int_0^2 \int_0^3 2e^{x+2y} \; dxdy$$

Method 2

$$\int_0^2\int_0^3 2e^{x+2y}dxdy$$

$$=\int_0^2\int_0^3 2e^x \cdot e^{2y}dxdy$$

$$=\int_0^2 e^{2y}\int_0^3 2e^x dx\, dy$$

$$=\int_0^2 e^{2y}\cdot(2e^3-2)\, dy$$

$$=(e^3-1)\int_0^2 e^{2y}d2y$$

$$=(e^3-1)\, e^{2y}\Big|_0^2 = (e^3-1)(e^4-1)$$

$$\int_0^2 \int_0^3 2e^{x+2y} \; dxdy$$

$$= \int_0^2 \int_0^3 2e^{x+2y}d(x+2y)\, dy$$

$$= \int_0^2 \left(2e^{x+2y}\Big|_0^3\right) dy$$

$$= \int_0^2 e^{2y+3} - e^{2y}\, d2y$$

$$= \int_0^2 e^{2y+3} d(2y+3) - \int_0^2 e^{2y} d2y$$

$$= e^{2y+3}\Big|_0^2 - e^{2y}\Big|_0^2$$

$$= e^7 - e^3 - e^4 + 1$$

Average Value

The average value of a function $f(x, y)$ over a rectangle R is defined to be

$$f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)dA$$

where $A(R)$ is the area of R.

Example 13. Find the average value of the paraboloid $f(x,y) = x^2 + y^2$ over $R = [-1,1] \times [-1,1]$.

$$f_{avg} = \frac{1}{A(R)} \int_{-1}^{1}\int_{-1}^{1} x^2+y^2 \, dx\, dy$$

$$= \frac{1}{4} \int_{-1}^{1} \frac{2}{3}+2y^2 \, dy$$

$$= \frac{1}{4} \left(\frac{2y}{3} + \frac{2}{3}y^3 \right)\Big|_{-1}^{1}$$

$$= \frac{1}{4}\left(\frac{4}{3}+\frac{4}{3}\right)$$

$$= \frac{2}{3}$$

$$\int_{-1}^{1} x^2+y^2 \, dx$$

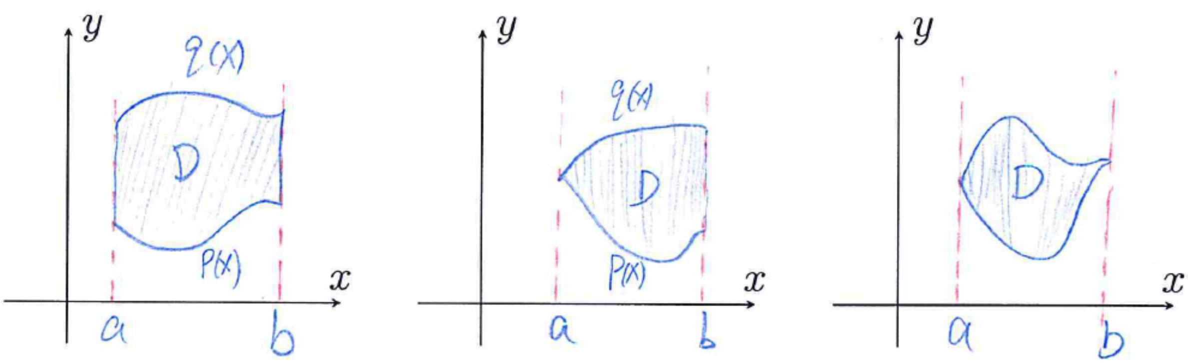
$$= \frac{x^3}{3} + xy^2 \Big|_{-1}^{1}$$

$$= \frac{2}{3} + 2y^2$$

Double Integrals over General Regions

• A plane region D is said to be of **type (I)** if it lies between the graphs of two continuous functions $p(x)$ and $q(x)$, that is,

$$D = \{(x,y) \mid a \le x \le b, p(x) \le y \le q(x)\}$$



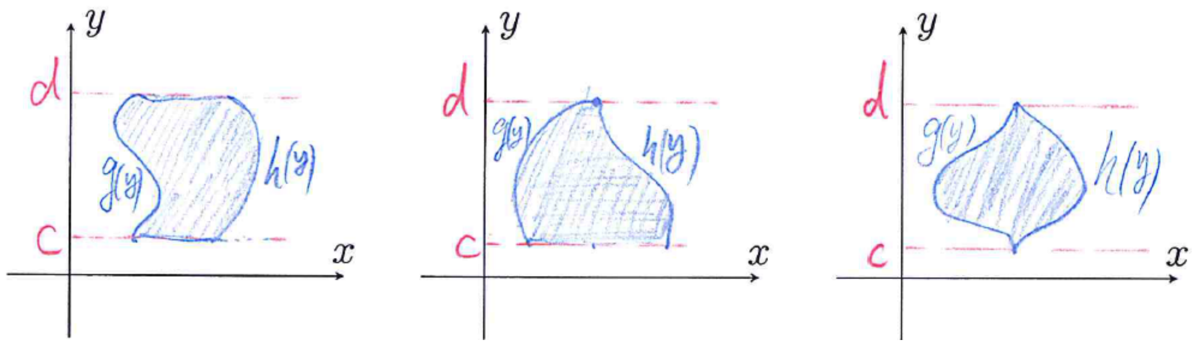
Theorem.

Suppose that $f = f(x,y)$ is a continuous function on D of type (I). the double integral can be calculated by iterated integrals:

$$\iint_D f(x,y)\, dA = \int_a^b \int_{p(x)}^{q(x)} f(x,y)\, dydx$$

• A plane region D is said to be of **type (II)** if it is given by

$$D = \{(x, y) \mid c \le y \le d, g(y) \le x \le h(y)\}$$



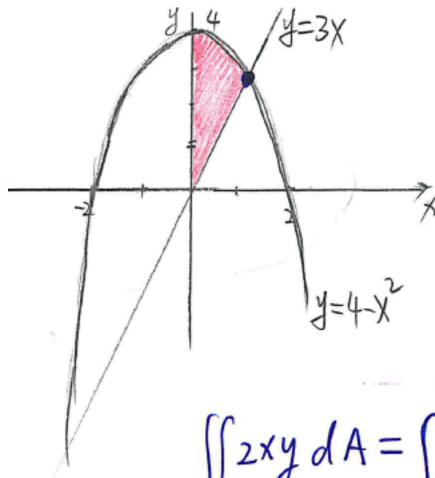
Theorem.

If D is the region of type (II), the double integral can be calculated by iterated integrals:

$$\iint_D f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dxdy$$

Example 14 (Type (I)). Let D the region in the **1st** quadrant bounded by $y = 4 - x^2$ and $y = 3x$.

Sketch the region D and calculate the double integral $\iint_D 2xy\, dA$.



Find the intersection point

$$\begin{cases} y = 4 - x^2 \\ y = 3x \end{cases} \Rightarrow 4 - x^2 = 3x$$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

$$x = -4 \quad x = 1$$

$$y = 3x = 3$$

$$\iint_D 2xy\, dA = \int_0^1 \int_{3x}^{4-x^2} 2xy\, dy\, dx$$

$$= \int_0^1 \left(xy^2 \Big|_{3x}^{4-x^2} \right) dx$$

$$= \int_0^1 x(4-x^2)^2 - x(3x)^2\, dx$$

$$= \int_0^1 x^5 - 8x^3 + 16x - 9x^3\, dx$$

$$= \int_0^1 x^5 - 17x^3 + 16x\, dx$$

$$= \frac{x^6}{6} - \frac{17}{4}x^4 + 8x^2 \Big|_0^1 = \frac{1}{6} - \frac{17}{4} + 8$$

Example 15 (Type (II)). Let D the region bounded by the graphs $x = y$ and $x = y^2$. Sketch the region D and calculate the double integral $\iint_D 2xy + 6y^2 \, dA$.

$$\iint_D 2xy + 6y^2 \, dA.$$

$$\int_0^1 \int_{y^2}^{y} (2xy + 6y^2)\, dx\, dy$$

$$= \int_0^1 (x^2 y + 6y^2 x)\Big|_{y^2}^{y} dy$$

$$= \int_0^1 (y^3 + 6y^3) - (y^5 + 6y^4)\, dy$$

$$= \int_0^1 -y^5 - 6y^4 + 7y^3\, dy$$

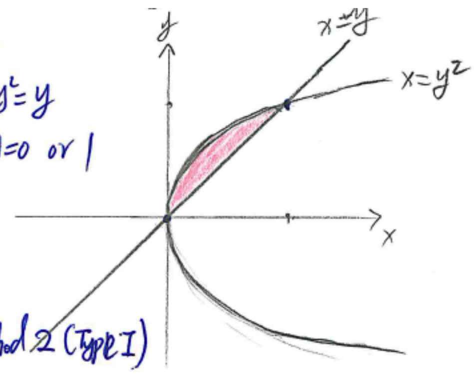
$$= -\frac{y^6}{6} - \frac{6y^5}{5} + \frac{7y^4}{4}\Big|_0^1$$

$$= -\frac{1}{6} - \frac{6}{5} + \frac{7}{4}$$

$$= \frac{23}{60}$$

$$\begin{cases} x = y \\ x = y^2 \end{cases} \Rightarrow y^2 = y$$

$y = 0$ or 1



Method 2 (Type I)

$$\int_0^1 \int_x^{\sqrt{x}} 2xy + 6y^2\, dy\, dx$$

$$= \int_0^1 xy^2 + 2y^3 \Big|_x^{x^{\frac{1}{2}}} dx$$

$$= \int_0^1 (x^2 + 2x^{\frac{3}{2}}) - (x^3 + 2x^3)\, dx$$

$$= \int_0^1 -3x^3 + x^2 + 2x^{\frac{3}{2}}\, dx$$

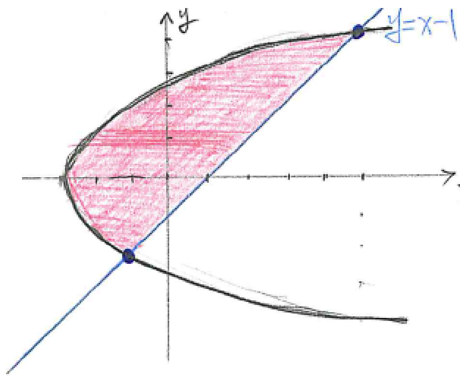
$$= -\frac{3}{4}x^4 + \frac{x^3}{3} + \frac{2x^{\frac{5}{2}}}{\frac{5}{2}}\Big|_0^1$$

$$= -\frac{3}{4} + \frac{1}{3} + \frac{4}{5}$$

$$= \frac{23}{60}$$

Example 16 (Type (II)). Let D the region bounded by the the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

Sketch the region D and calculate the double integral $\iint_D 2xy \, dA$.



Find intersection points

$$\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases}$$

$y^2 = 2(y+1) + 6$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = 4 \quad y = -2$

$$\iint_D 2xy \, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2 - 3}^{y+1} 2xy \, dx \, dy$$

$$= \int_{-2}^{4} x^2 y \Big|_{\frac{1}{2}y^2 - 3}^{y+1} dy$$

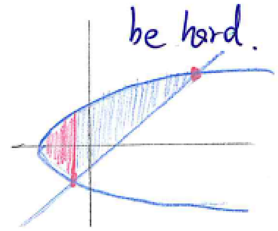
$$= \int_{-2}^{4} y\left((y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right) dy$$

$$= \int_{-2}^{4} -\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \, dy$$

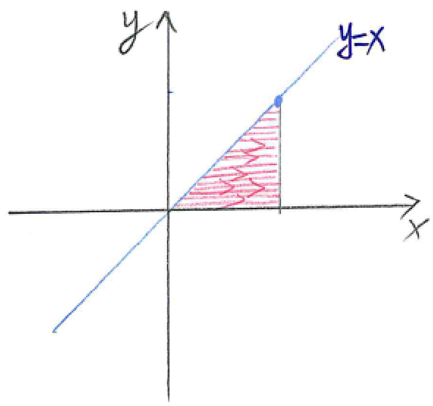
$$= -\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \Big|_{-2}^{4}$$

$$= 72$$

Using Type (I) will be hard.



Example 17. Evaluate the iterated integral $\int_0^1 \int_y^1 e^{-x^2}\, dxdy$.



$D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le y \le 1 \\ y \le x \le 1 \end{array} \right\}$

hard to do directly.

$$\int_0^1 \int_y^1 e^{-x^2} dx\, dy = \iint_D e^{-x^2} dA = \int_0^1 \int_0^x e^{-x^2} dy\, dx$$

$$= \int_0^1 y e^{-x^2} \Big|_0^x \, dx$$

$$= \int_0^1 x e^{-x^2} dx$$

$$= -\frac{1}{2}\int_0^1 e^{-x^2} d(-x^2)$$

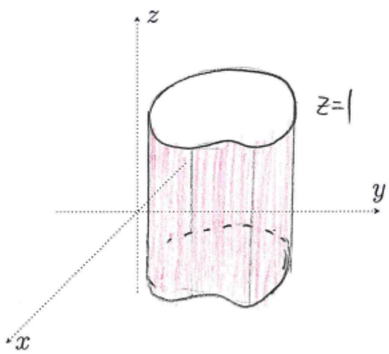
$$= -\frac{1}{2} e^{-x^2} \Big|_0^1$$

$$= -\frac{1}{2}\left(e^{-1} - 1\right)$$

Theorem.

Area of the region D can by computed by

$$\text{Area of } D = \iint_D 1 \, dA$$



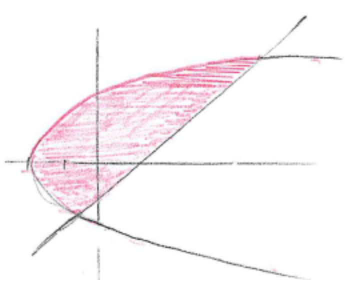
volume with heigh 1

$V = \text{Area} \cdot 1$

Example 18. Let D be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Find the area of D.

$$\text{Area} = \iint_D dA$$

$$= \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} dx\,dy = \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy$$

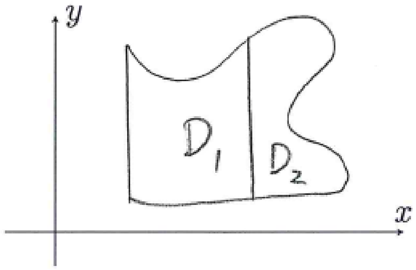


$$= \frac{y^2}{2} + y - \frac{y^3}{6} + 3y \Big|_{-2}^{4}$$

$$= \frac{y^2}{2} - \frac{y^3}{6} + 4y \Big|_{-2}^{4} = 6$$

Review: 

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$



$D = D_1 \cup D_2$

D_1 type (I), D_2 Type (II)

Property of the double integrals :

Theorem.

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_1} f(x,y)dA$$