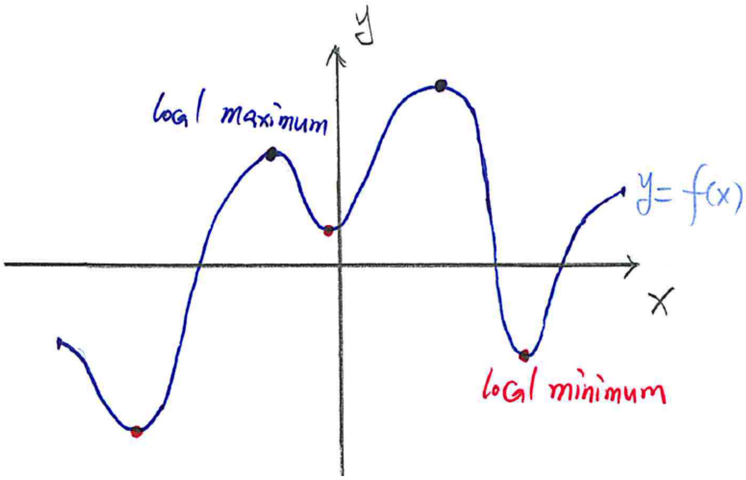
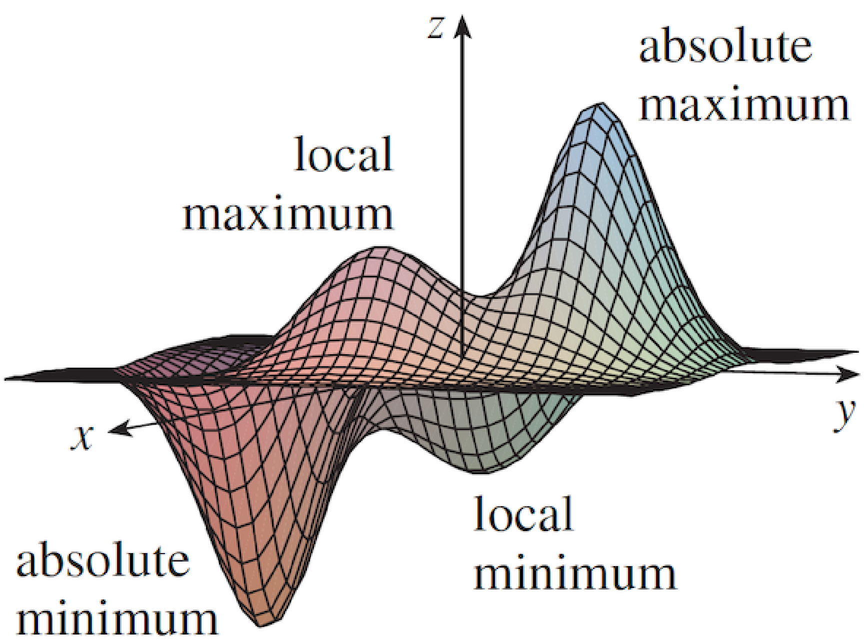
§2.9 Maximum and Minimum Values

Single variable function $z = f(x)$



Two variables function $z = f(x, y)$



Definition.

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a **local maximum value** (maxima).

A function of two variables has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a **local minimum value** (minima).

Theorem.

If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

A point (a,b) is called a **critical point** of f if $f_x(a,b)=0$ and $f_y(a,b)=0$, or if one of these partial derivatives does not exist.

Example 1. Let $f(x,y)=x^2+y^2-4x-2y+18$

$f_x = 2x-4=0 \Rightarrow x=2$

$f_y = 2y-2=0 \Rightarrow y=1$

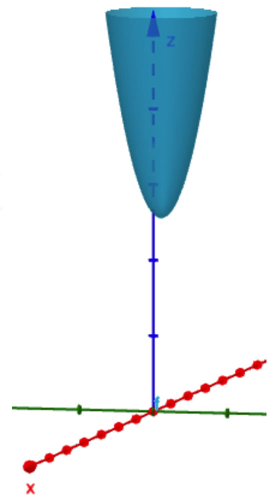
So $(2,1)$ is a critical point.

completing the square $f(x,y)=(x-2)^2+(y-1)^2+13$

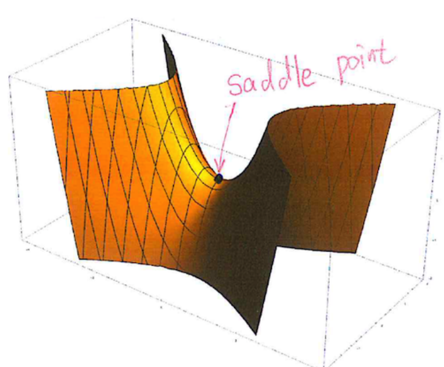
$f(2,1)=13$

and $f(x,y)\geq 13$ for all values near $(2,1)$

So $(2,1,13)$ is a local minimum point.



Example 2. Find the extreme values of $f(x,y)=y^2-x^2$



$f_x = -2x=0 \Rightarrow x=0$

$f_y = 2y=0 \Rightarrow y=0$

The critical point is $(0,0)$.

$f(0,0)=0$

Along x-axis $(y=0)$, $f(x,0)=-x^2\leq 0$

Along y-axis $(x=0)$, $f(0,y)=y^2\geq 0$.

So $(0,0)$ is not a maximum or minimum.

Theorem. The Second Derivative Test:

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$, [that is, (a,b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(1) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

(2) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

(3) If $D<0$, then $f(a,b)$ is not a local maximum or minimum

• If $D=0$, the test gives **NO** information.

• In case (3) the point is called a **saddle** point of f and the graph of f crosses its tangent plane at (a,b).

•To memorize D, we use determinant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Example 3. Let $f(x,y) = x^2+y^2-4x-2y+18$

$f_{xx}=2$

$f_{yy}=2$

$f_{xy}=0=f_{yx}$

$D(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$D(2,1)=4>0$

$f_{xx}=2>0$

So $f(2,1)$ is a local minimum point.

Example 4. Find the extreme values of $f(x,y)=y^2-x^2$

$f_{xx}=-2$

$f_{yy}=2$

$f_{xy}=f_{yx}=0$

$D(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0$

$f(0,0)$ is a saddle point.

Example 5. Find the local maximum and minimum values and saddle points of

$$f(x,y)=x^3+y^3-3xy+4$$

$f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2$

$f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2$

$x = (x^2)^2 = x^4 \Rightarrow x^4 - x = 0 \Rightarrow x(x^3-1) = 0$

$\Rightarrow x(x-1)(x^2+x+1) = 0$

$\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} x=1 \\ y=1 \end{cases}$

$f_{xx} = 6x$

$f_{yy} = 6y$ $D(x,y) = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$

$f_{xy} = f_{yx} = -3$

For $(0,0)$, $D(0,0) = -9 < 0$ implies $f(0,0)$ is a saddle point.
For $(1,1)$, $D(1,1) = 27$ and $f_{xx}(1,1) = 6 > 0$ implies $f(1,1) = 3$ is a local minimum.

Example 6. Find and classify the critical points of the function

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 7$$

Also find the highest point on the graph of f.

$f_x = 3x^2 + 6x = 0$ implies $x^2 + 2x = 0$ implies $x(x+2=0)$. So $x = 0$, or $x = -2$.
$f_y = 3y^2 - 6y = 0$ implies $y^2 - 2y = 0$ implies $y(y-2) = 0$. So $y = 0$, or $y = -2$.
All critical points are $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$.

$f_{xx} = 6x+6$

$f_{yy} = 6y-6$ $D(x,y) = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = (6x+6)(6y-6) - 0$

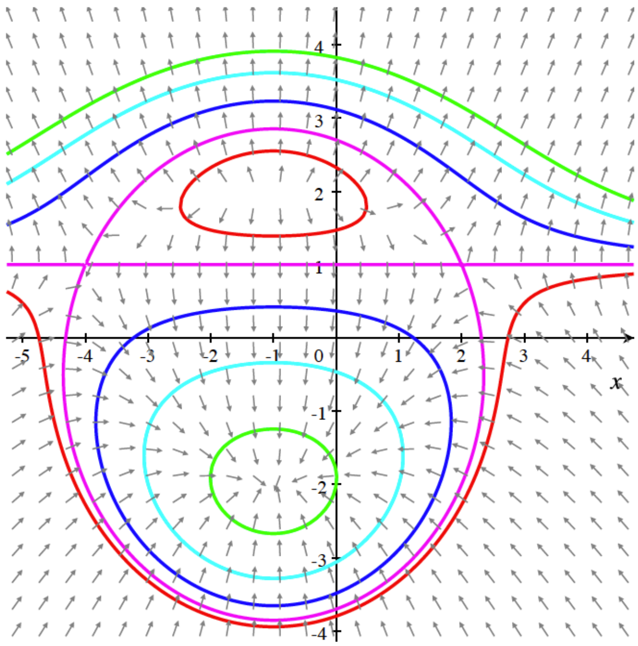
$f_{xy} = f_{yx} = 0$ $= 36(x+1)(y-1)$

- $(0,0)$: $D(0,0) = -36 < 0$. $\Rightarrow f(0,0) = -7$ is a saddle point
- $(0,2)$: $D(0,2) = 36 > 0$, $f_{xx}(0,2) = 6 > 0$ $\Rightarrow f(0,2) = -12$ is a local minimum point.
- $(-2,0)$: $D(-2,0) = 36 > 0$, $f_{xx}(-2,0) = -6 < 0$ $\Rightarrow f(-2,0) = -4$ is a local maximum.
- $(-2,2)$: $D(-2,2) = -36 < 0$ $\Rightarrow f(-2,2) = -8$ is a saddle point

If we only know the gradients vectors and level curve, we can also determine the local maximum and minimum values, using the property that"the gradient vectors are in directions in which the value of $f(x, y)$ increases rapidly. " (§2.5)

The intersection points give the saddle points.

HW20 Suppose that the values of $f(x, y)$ for the given level curves are -2, -1, 0, 1, and 2. The same color has the same level curve.



HW35 $f(x, y) = x^2 + y^2(1-x)^3$ only has one critical point $(0, 0)$. $f(0, 0) = 0$. By second derivative test, it is a local minimum.

However, the $f(x, y)$ can approach any small number. For example, $f(3, 3) = -63$.

§2.10 Optimization

Theorem. Extreme Value Theorem

If f is continuous on a closed, bounded set (domain) D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

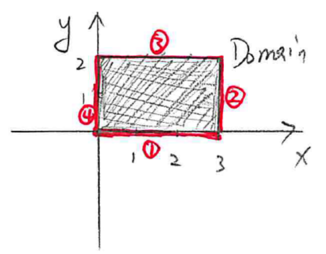
• To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

Step 1. Find the values of f at the critical points of f in D.

Step 2. Find the extreme values of f on the boundary of D.

Step 3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 1. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.



step 1

• $f_x = 2x - 2y = 0 \Rightarrow x = y$

• $f_y = -2x + 2 = 0 \Rightarrow x = 1$

• Critical point $(1, 1)$. $f(1,1) = 1$

step 2:

1 $y = 0$, $0 \le x \le 3$. $f(x, 0) = x^2$. $\Rightarrow 0 \le f(x, 0) \le 9$

2 $x = 3$ $0 \le y \le 2$ $f(3, y) = 9 - 6y + 2y = 9 - 4y \Rightarrow 1 \le f(3, y) \le 9$

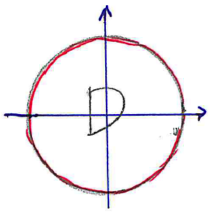
3 $y = 2$ $0 \le x \le 3$ $f(x, 2) = x^2 - 4x + 4 = (x-2)^2 \Rightarrow 0 \le f(x, 2) \le 4$

4 $x = 0$ $0 \le y \le 2$ $f(0, y) = 2y \Rightarrow 0 \le f(0, y) \le 4$

Step 3. Absolute maximum is $f(3, 0) = 9$

Absolute minimum is $f(0, 0) = 0$.

Example 2. Find the extreme values of $f(x, y) = x^2 + 2y^2 + 2$ on the disc D equation $x^2 + y^2 \le 1$. (We can think the background as temperature function $f(x, y)$ of a plate D)



Step 1. $\nabla f(x,y) = \langle 2x, 4y\rangle$. So the critical point is $(0,0)$. The local extreme value is $f(0,0) = 2$

Step 2. The boundary of D is the circle $x^2 + y^2 = 1$. On the boundary, $f(x,y) = x^2 + 2y^2 + 2 = y^2 + 3$. We know that on $x^2 + y^2 = 1$, $0 \le y^2 \le 1$. So, $3 \le f(x,y) \le 4$.

Step 3. The maxima and mimima of $f(x,y)$ on the disc D are $f(0,\pm 1) = 4$ and $f(0,0) = 2$

Example 3. A rectangular box is to be made from cardboard with no top and volume 4 m^3. Find the dimension of the box to minimize the amount of cardboard.

Suppose the length, width, and height of the box are x, y, z. So $xyz = 4$.
The total area of the cardboard is $f(x,y,z) = xy + 2yz + 2xz$.
So, with constraint $xyz = 4$, we have $f = xy + 8x^{-1} + 8y^{-1}$.
To find the critical points, $\nabla f(x,y) = \langle y - 8x^{-2}, x - 8y^{-2}\rangle = \vec{0}$.
Solve the equations, we have $x = y = 2$ and by $xyz = 4$ we have $z = 1$.