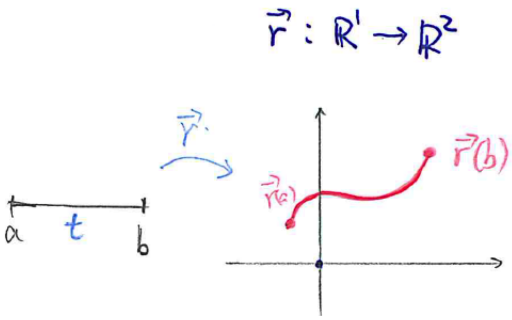
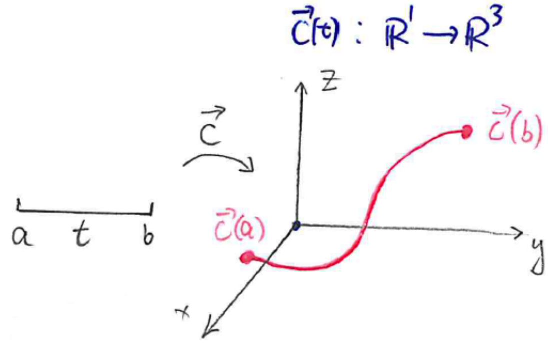
§2.8 Parameterizing surfaces

Review of §1: We can use parametric equation $\vec{r}(t) = \langle x(t), y(t)\rangle$ or $\vec{c}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$ to describe **curves** in $\mathbb{R}^2$ or $\mathbb{R}^3$.

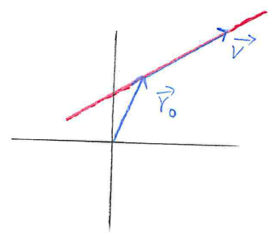




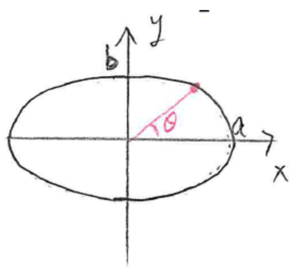
Example 1. Line in $\mathbb{R}^2$ or $\mathbb{R}^3$.

$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle a, b\rangle + t\langle c, d\rangle$.

Then $x(t) = a + ct$, $y(t) = b + ct$.



Example 2. Ellipse in $\mathbb{R}^2$.

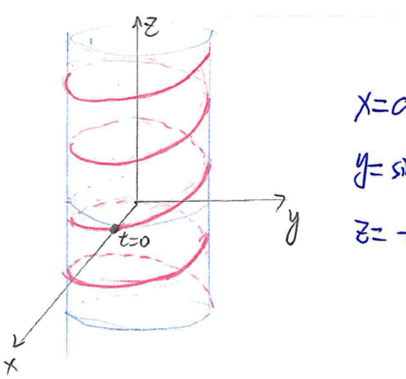


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{cases} \frac{x}{a} = \cos\theta \\ \frac{y}{b} = \sin\theta \end{cases} \quad 0 \le \theta \le 2\pi$$

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases} \quad 0 \le \theta \le 2\pi$$

Example 3. Helix in $\mathbb{R}^3$.



$x = \cos t$

$y = \sin t$

$z = t$

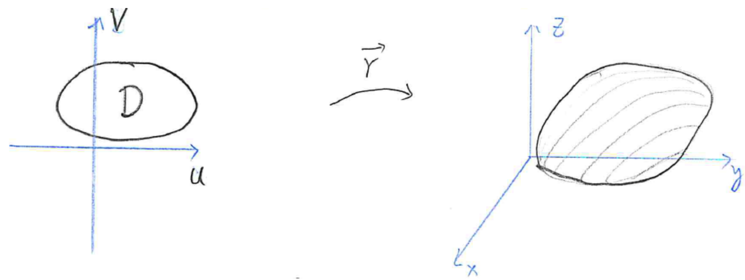
We know that $z = f(x, y)$ or $F(x, y, z) = 0$ describe a surface in $\mathbb{R}^3$.

We use a vector function $\vec{r}(u, v)$ with two parameters u, v to describe **surface in** $\mathbb{R}^3$.

$$\begin{aligned}\vec{r}(u,v) &= x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}\\ &= \langle x(u,v),\ y(u,v),\ z(u,v)\rangle\end{aligned}$$

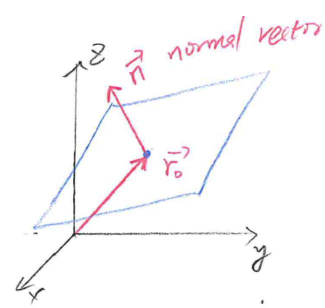
This means that the surface contains all points $(x, y, z) \in \mathbb{R}^3$ such that

$$x = x(u,v),\ y = y(u,v),\ z = z(u,v).$$



Example 4. Give a parametric description for a plane in $\mathbb{R}^3$.

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$
$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$
$ax + by + cz = d$
So, we can use $y = u$, $z = v$ as parameters
to describe the equation
$x = \dfrac{d - bu - cv}{a}$, $y = u$, $z = v$.



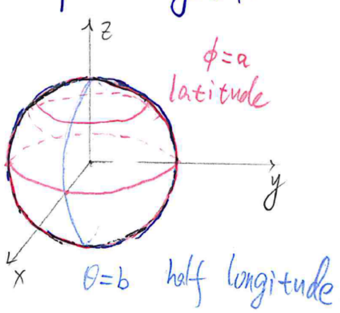
Example 5. Give a parametric description for a Sphere $x^2 + y^2 + z^2 = r^2$ in $\mathbb{R}^3$.

Spherical system:

$x = r\sin\phi\cos\theta$ $\quad 0 \le \phi \le \pi$

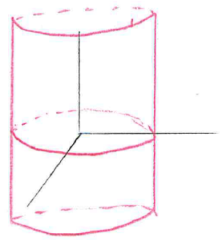
$y = r\sin\phi\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$z = r\cos\phi$



Example 6. Give a parametric description for a Cylinder in $\mathbb{R}^3$.

A cylinder can be described as $f(x, y) = 0$. So, we can get $x = g(u)$, $y = h(u)$ and $z = v$.

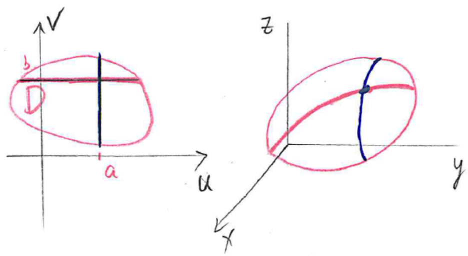


$x = a\cos u$

$y = b\sin u$

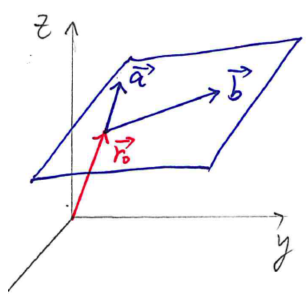
$z = v$

Grid curves: (1) Put $u = a$. (b) Put $v = b$.



Example 7. (**Parametrize a Plane**) Plane in $\mathbb{R}^3$ with position vector $\vec{r}_0$ such that contains two vectors $\vec{a}$ and $\vec{b}$.

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.



$\vec{r} = \vec{r}_0 + u\vec{a} + v\vec{b}$

$x(u,v) = x_0 + ua_1 + vb_1$

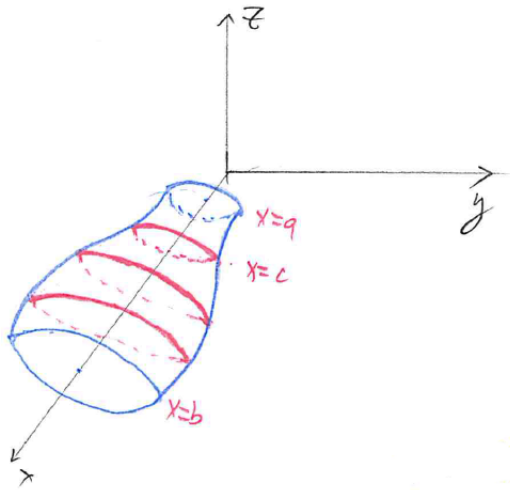
$y(u,v) = y_0 + ua_2 + vb_2$

$z(u,v) = z_0 + ua_3 + vb_3$

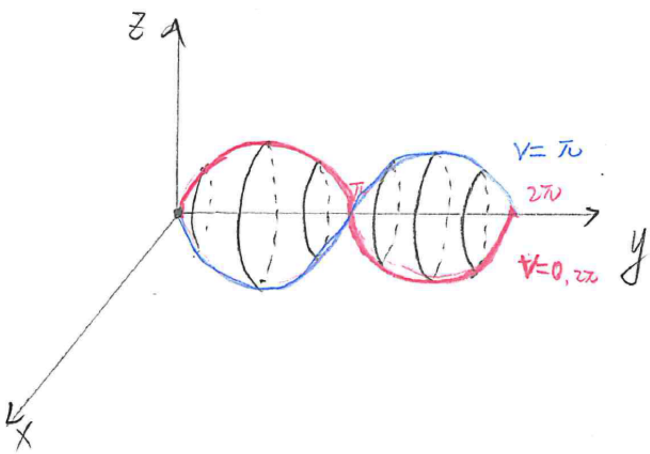
Surfaces of Revolution

$$x = x, \quad y = f(x)\cos\theta, \quad z = f(x)\sin\theta$$

for $a \le x \le b$, $f(x) \ge 0$ and $0 \le \theta \le 2\pi$.



Example 8. Describe $x = \sin u \sin v$, $\quad y = u$, $\quad z = \sin u \cos v$ for $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.



Tangent Planes. Suppose the surface is given by $F(x, y, z) = 0$.

In §2.1, the tangent plane to the level surface at point $P(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Now, suppose **parametric surface** is given by $\vec{r}(u, v) = \langle x, y, z \rangle$,

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v).$$

Two **tangent vectors** at $\vec{r}(u_0, v_0) = (x_0, y_0, z_0)$ are given by

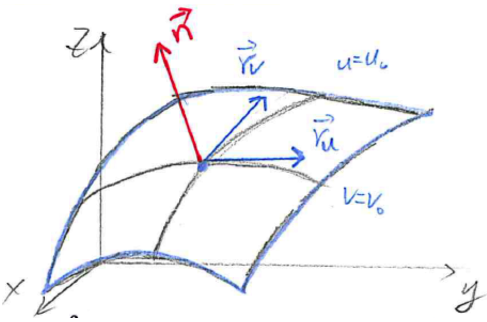
$$\vec{r}_u(u_0, v_0) = \langle x_u(u_0, v_0), y_u(u_0, v_0), z_u(u_0, v_0) \rangle$$
$$\vec{r}_v(u_0, v_0) = \langle x_v(u_0, v_0), y_v(u_0, v_0), z_v(u_0, v_0) \rangle$$

The tangent plane has a parametric description:

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + a\vec{r}_u + b\vec{r}_v$$

The **normal vector** is given by

$$\vec{n} = \vec{r}_u \times \vec{r}_v.$$



A continuously differentiable parameterization (parametric surface) $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ is called **regular** at (u_0, v_0) if neither $\vec{r}_u$ nor $\vec{r}_v$ is a scalar multiple of the other at (u_0, v_0).

$\vec{r}_u$ and $\vec{r}_v$ are not scalar multiple of each other at (u_0, v_0) if and only if $\vec{r}_u \times \vec{r}_v \neq 0$. In this case, $\vec{r}_u$ and $\vec{r}_v$ are called linearly independent, (as in Linear Algebra.)

A continuously differentiable parameterization $\vec{r}(u, v)$ is called a **local regular parameterization** if it is regular on all points in domain.

Example 9. Is $\vec{r}(u, v) = \langle u, u \cos v, u \sin v \rangle$ a local regular parametrization?

$\vec{r}_u = \langle 1, \cos v, \sin v \rangle$ and $\vec{r}_v = \langle 0, -u \sin v, u \cos v \rangle$. When $u = 0$, $\vec{r}_v = 0\vec{r}_u$. $\vec{r}(u, v)$ is not regular on points $(0, v)$. So, it is not a local regular parametrization.

Example 10. Is $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$ a local regular parametrization for a continuous differentiable f?

$\vec{r}_u = \langle 1, 0, f_u \rangle$ and $\vec{r}_v = \langle 0, 1, f_v \rangle$. So, $\vec{r}_u$ and $\vec{r}_v$ are always independent. So, it is a local regular parametrization.

Example 11. Suppose the surface is given by parametric equations $x = 2u$, $y = uv$, $z = v^2$. Find the tangent plane to the surface at $(2, 1, 1)$.

Which point is not regular?

tangent vectors: $u=1$ $v=1$

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle = \langle 2, v, 0 \rangle$$

$$\vec{r}_v = \langle x_v, y_v, z_v \rangle = \langle 0, u, 2v \rangle$$

normal vector

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & v & 0 \\ 0 & u & 2v \end{vmatrix}$$

$$= 2v^2 \, \vec{i} - 4v \, \vec{j} + 2u \, \vec{k}$$

at $(2,1,1)$, the normal vector is

$$\vec{n} = \langle 2, -4, 2 \rangle$$

The tangent plane at $(2,1,1)$ is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle 2, -4, 2 \rangle \langle x-2, y-1, z-1 \rangle = 0$$

$$2(x-2) - 4(y-1) + 2(z-1) = 0$$

$$2x - 4y + 2z - 2 = 0$$

or tangent parametric plane:

$$\vec{r} = \vec{r}_0 + u\vec{a} + v\vec{b}$$

Here $\vec{a} = \langle 2, 1, 0 \rangle$

$\vec{b} = \langle 0, 1, 2 \rangle$

$\vec{r}_0 = \langle 2, 1, 1 \rangle$

$$\begin{cases} x = 2 + 2v \\ y = 1 + u + v \\ z = 1 + 2u \end{cases}$$

The only non-regular point is when $v = 0$ and $u = 0$. So, it is $(0, 0, 0)$.