§2.5 Directional derivatives

Recall the partial derivative of $f(x, y)$ with respect to x at (x_0, y_0)

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

is the slope of the tangent line at (x_0, y_0) to the trace in the plane $y = y_0$, that is, in the direction of the unit vector $\vec{i} = \langle 1, 0 \rangle$. Similarly, the partial derivative

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

is the slope of the tangent line at (x_0, y_0) to the trace in the plane $y = x_0$, that is, in the direction of the unit vector $\vec{j} = \langle 0, 1 \rangle$.

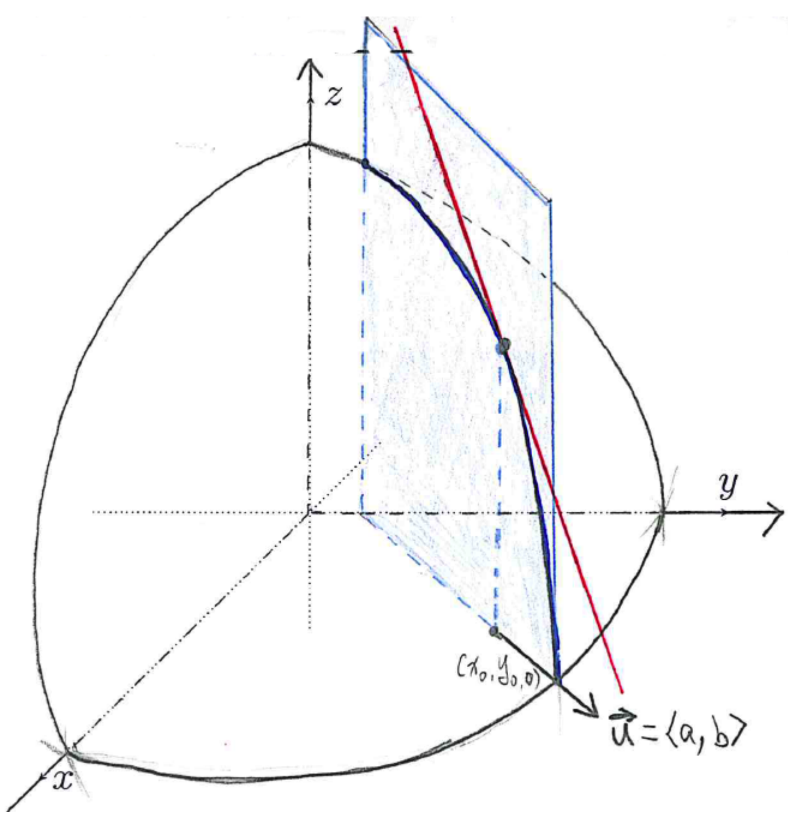
Definition.

The **directional derivative** of f at (x_0, y_0) in the direction of a ***unit*** vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

We already see that $D_{\vec{i}} f = f_x$ and $D_{\vec{j}} f = f_y$.

Geometrically, the directional derivative $D_{\vec{u}} f(x_0, y_0)$ is the **rate of change** of f at (x_0, y_0) in the direction of $\vec{u}$. It is the **slope** of the tangent line.



The next theorem provides an easier method for computing the directional derivative.

Theorem.

A differentiable function f has a directional derivative in the direction of any **unit** vector $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = (\nabla f) \cdot \vec{u}.$$

Proof.

: Define a function g of variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

Then by definition

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0)$$

by chain rule

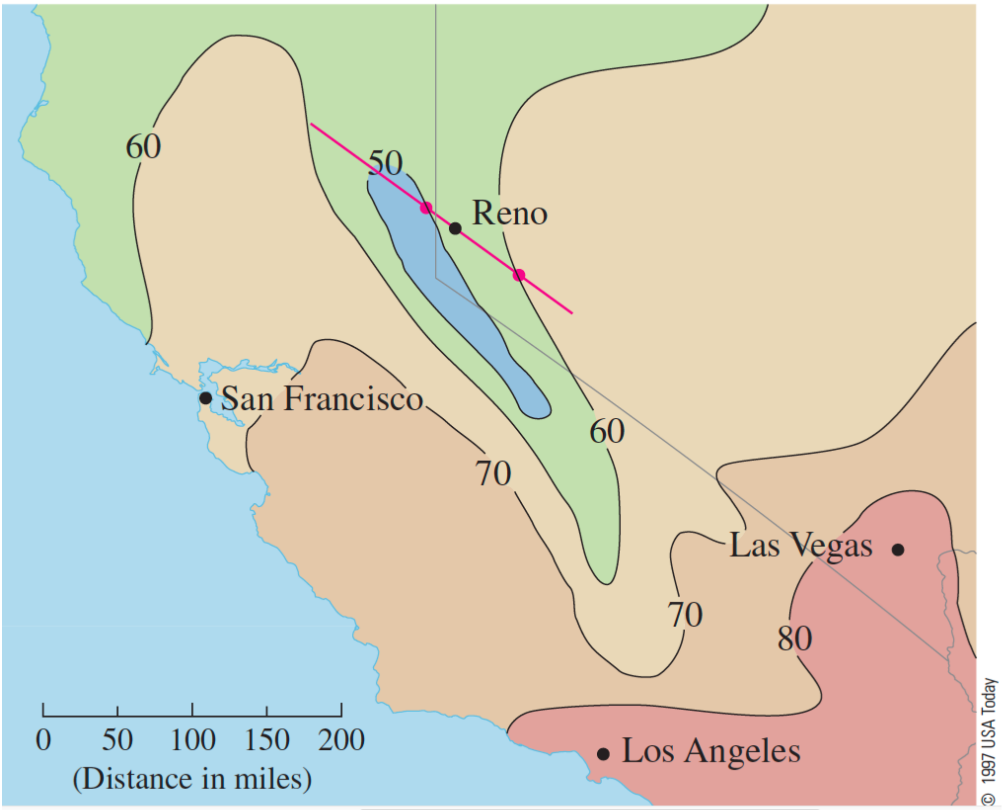
$$g'(h) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Then $D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$

Example 1. Estimate the value of directional derivative of the temperature function at Reno in the Southeast direction $\vec{u} = \frac{1}{\sqrt{2}}(\vec{i} - \vec{j})$

Look at the following Contour map or Level curves.



$$D_{\vec{u}}T \approx \frac{60-50}{75} = 10/75 \approx 0.13°F/mile$$

Example 2. Find the directional derivative $D_{\vec{u}}f(x,y)$ if

$$f(x,y) = x^2 + 3xy - 2y^4$$

and $\vec{u}$ is the unit vector in the direction of $\langle 1, 2\rangle$. What is $D_{\vec{u}}f(0,1)$?

$$\vec{u} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

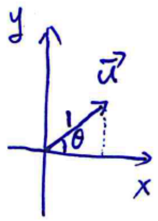
$$D_{\vec{u}} f(x,y) = f_x \cdot a + f_y \cdot b = (2x+3y)\frac{1}{\sqrt{5}} + (3x - 8y^3)\frac{2}{\sqrt{5}}$$

$$D_{\vec{u}} f(0,1) = \frac{3}{\sqrt{5}} + (-8)\frac{2}{\sqrt{5}} = -\frac{13}{\sqrt{5}}$$

Example 3. Find the directional derivative $D_{\vec{u}}f(x,y)$ if

$$f(x,y) = x^2 + 3xy - 2y^4$$

and $\vec{u}$ is the unit vector making $\theta = \pi/3$ with positive x-axis. What is $D_{\vec{u}}f(1,0)$?



$\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$D_{\vec{u}} f(x,y) = f_x \cdot a + f_y \cdot b = \frac{1}{2}(2x+3y) + \frac{\sqrt{3}}{2}(3x-8y^3)$

$D_{\vec{u}} f(1,0) = \frac{1}{2}(2) + \frac{\sqrt{3}}{2}(3) = 1 + \frac{3\sqrt{3}}{2}$

Maximizing the direction derivative

Suppose f is a differentiable function of two or three variables.

Question: In which of these directions does change fastest (rate) and what is the maximum rate of change?

Theorem.

The maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta = |\nabla f| \cos\theta$

So, when $\theta = 0$, $D_{\vec{u}} f$ has the maximum value, $|\nabla f|$.
$\cos\theta = 1$

Example 4. Suppose the temperature at a point (x, y, z) is given by $T(x, y, z) = xe^y + 0.5z^2$. In which direction does the temperature increase fastest at the point $(1, 0, 3)$? What is the maximum rate of increase?

$\nabla T = \langle e^y, xe^y, z \rangle$

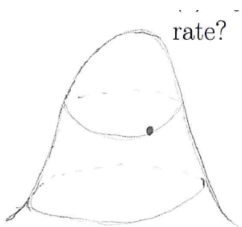
$\nabla T(1,0,3) = \langle 1, 1, 3 \rangle$

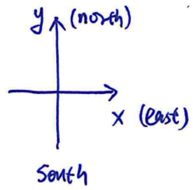
- In the direction $\langle 1, 1, 3 \rangle$ the temperature increase fastest at $(1,0,3)$
- The maximum rate of increase is $|\nabla T(1,0,3)| = \sqrt{1+1+9} = \sqrt{11}$.

Example 5. Suppose you are climbing a hill whose shape is given by the equation $z = 1100 -$

$2x^2 - y^2$, where x, y, and z are measured in meters, and you are standing at a point with coordinates $(10, 20, 500)$. The positive x-axis ($\vec{i}$) points east and the positive y-axis ($\vec{j}$) points north.

(1) If you walk due south, will you start to ascend or descend? At what rate?





$\vec{u} = \langle 0, -1 \rangle$

$$D_{\vec{u}} z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle \cdot \langle 0, -1 \rangle$$

$$= \langle -4x, -2y \rangle \cdot \langle 0, -1 \rangle = 2y = 40$$

Ascend at rate 40

(2) In which direction is the slope largest? What is the greatest rate of clime.

$$\nabla z = \langle -4x, -2y \rangle$$

$$= \langle -40, -40 \rangle$$

- In the direction of $\langle -40, 40 \rangle$ the slope is the largest
- The greatest rate is

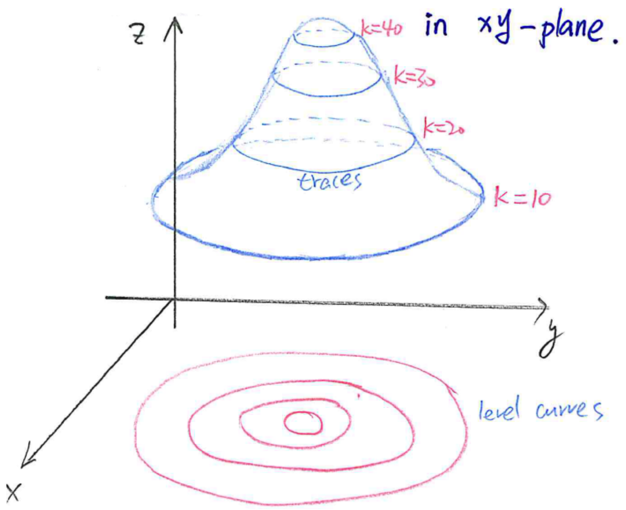
$$|\nabla z| = \sqrt{40^2 + 40^2} = 40\sqrt{2}.$$

§2.7 Level Sets and Gradient Vector

Recall from §2.0 for level curves.

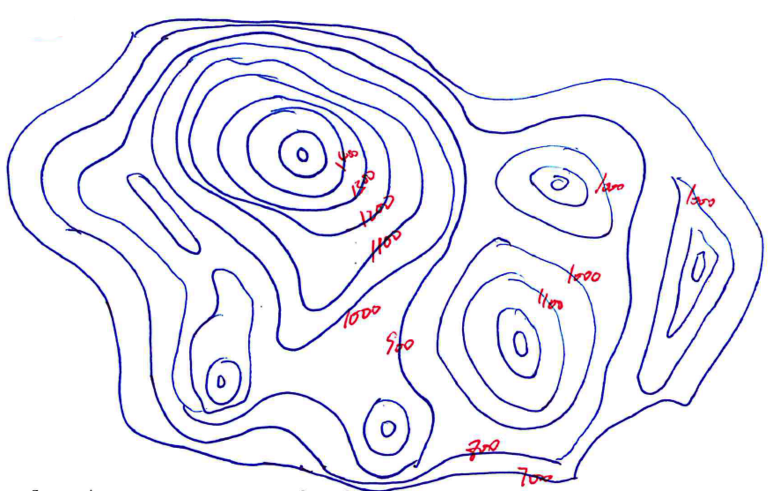
Definition.

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = c$, where is c a constant (in the range of f).



The level curves $f(x, y) = c$ are just the traces of the graph of f in the horizontal plane $z = c$ projected down to the xy-plane.

One common example of level curves (contour map) occurs in topographic maps of mountainous regions.



More generally, for a function $f(x, y, z)$ with three variables or $f(x_1, x_2, \ldots, x_n)$, the **level set** is $f(x, y, z) = c$.

For a point $P = (a, b, c)$, the level set of $f(x, y, z)$ containing P is $f(x, y, z) = f(a, b, c)$.

The formula for the directional derivative in the direction of $\vec{u} = \langle a, b\rangle$

$$D_{\vec{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \vec{u}$$

Definition.

The **gradient** of a function $f(x,y)$ is the vector function ∇f by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

The notation ∇ is pronounced Del or nabla.

The formula for the directional derivative in the direction of $\vec{u} = \langle a, b\rangle$ is

$$D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

or $D_{\vec{u}}f = \nabla f \cdot \vec{u}$ for short.

Example 1. (1) Find the gradient of $f(x,y) = \sin(xy) + e^y$ at $(0,1)$.

$\nabla f(x,y) = \langle y\cos(xy), x\cos(xy) + e^y\rangle$

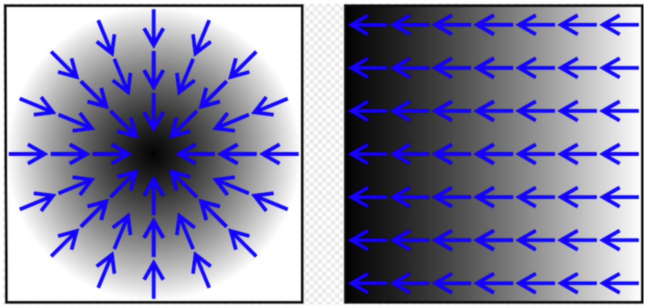
$\nabla f(0,1) = \langle 1, 3\rangle$

(2) Find the directional derivative of $f(x,y)$ at $(0,1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 3\vec{j}$.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{4+9}}\langle 2, 3\rangle = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle$$

$$D_{\vec{u}}f = \langle 1, 3\rangle \cdot \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle = \frac{2}{\sqrt{13}} + \frac{3e}{\sqrt{13}} = \frac{2+3e}{\sqrt{13}}$$

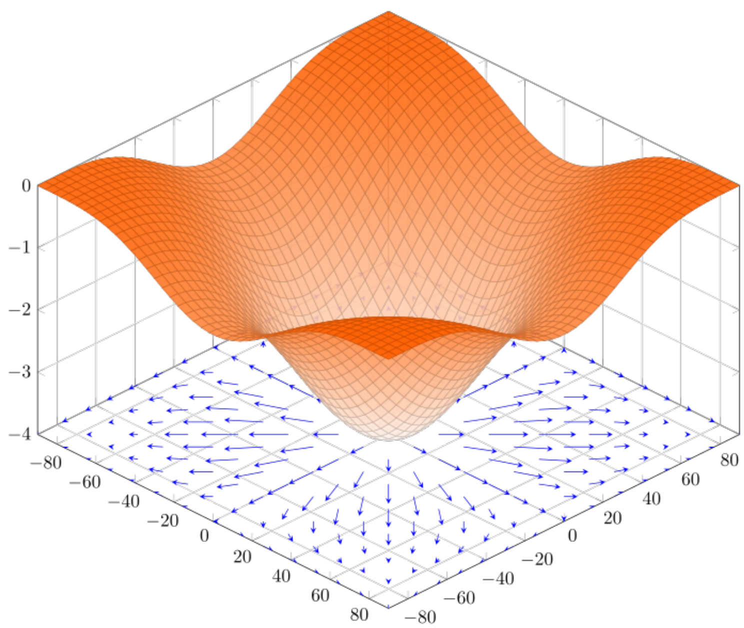
Example 2.



The first picture is the gradient $\nabla f(x,y) = \langle -\sin x, -\sin y\rangle$ for $f(x,y) = \cos x + \cos y$.

The second picture is the gradient $\nabla f(x,y) = \langle -1, 0\rangle$ for $f(x,y) = -x + 2$.

Example 3.



$f(x,y) = -(\cos^2 x + \cos^2 y)^2$

The gradient $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle -2(\cos^2 x + \cos^2 y)2\cos x(-\sin x), -2(\cos^2 x + \cos^2 y)2\cos y(-\sin y)\rangle$

- Functions of three variables $f(x,y,z)$:

Definition.

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a *unit* vector $\vec{u} = \langle a, b, c\rangle$ is

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

Using vector notation, denote $\vec{x}_0 = \langle x_0, y_0, z_0\rangle$, we have

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

Definition.

The **gradient** of $f(x,y,z)$ is the vector function ∇f by

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

The formula for the directional derivative in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u} = f_x a + f_y b + f_z c$$

Example 4. Let $f(x, y, z) = xyz$. (1) Find the gradient of f.

$\nabla f = \langle yz, xz, xy \rangle$

(2) Find the direction derivative of f at $(1, 2, 0)$ in the direction $\vec{v} = \langle -1, 2, 1 \rangle$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{6}} \langle -1, 2, 1 \rangle$.
$D_{\vec{u}} f(1, 2, 0) = \langle 0, 0, 2 \rangle \cdot \dfrac{1}{\sqrt{6}} \langle -1, 2, 1 \rangle = \dfrac{2}{\sqrt{6}}$

Tangent planes to level surfaces

Suppose S is a level surface with equation

$$F(x, y, z) = k$$

of three variables, and let $P(x_0, y_0, z_0)$ be a point on S.

Any curve C is described by a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let $t = t_0$ be the parameter value for P.

Theorem.

The gradient vector $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}\,'(t_0)$ to any curve C on S that passes through P.

The theorem implies that $\nabla F(x_0, y_0, z_0)$ is the normal vector of the tangent plane.

Proof of the theorem: Apply $\dfrac{\partial}{\partial t}$ to $F(x, y, z) = k$ and use the chain rule

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

So, $\nabla F \cdot \vec{r}'(t) = 0$.
At $t = t_0$, that is (x_0, y_0, z_0), we have $\nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$.
So, $\nabla F(x_0, y_0, z_0)$ is perpendicular to $\vec{r}'(t_0)$.

Definition.

The **tangent plane** to the level surface at point P is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Definition.

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. Its symmetric equation is

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example 5. Find the equations of the tangent plane and normal line at the point $(3, 2, 2)$ to the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 6$$

$F_x = \frac{2x}{9}$ $F_x(3,2,2) = \frac{2}{3}$

$F_y = \frac{2y}{4}$ $F_y(3,2,2) = 1$

$F_z = 2z$ $F_z(3,2,2) = 4$

The tangent plane : $\frac{2}{3}(x-3) + (y-2) + 4(z-2) = 0$

The normal line : $\frac{x-3}{2/3} = \frac{y-2}{1} = \frac{z-2}{4}$

HW18. Consider the function $F(x, y, z) = x^2 + y^2 + z^2$ and point $P = (1, 2, 3)$. (a) Find the level surface on point P. (b) Find the gradient vector ∇F at P. (c) Find an equation for the tangent plane to the level surface at the point P. (d) Put level surface $F = F(P)$, point P and tangent plane on https://www.geogebra.org/3d

(a)$F(1, 2, 3) = 14$. The level surface on P is $x^2 + y^2 + z^2 = 14$.
(b)The gradient vector $\nabla F = \langle 2x, 2y, 2z \rangle$. So $\nabla F(1, 2, 3) = \langle 2, 4, 6 \rangle$.
(c)The tangent plane to the level surface at P is
$2(x - 1) + 4(y - 2) + 6(z - 3) = 0$