§2.4 Differentiation rules

Theorem. Linear Property:

Let $f(x,y)$ and $g(x,y)$ be two differentiable functions and k a real number. Then we have

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial}{\partial x}f + \frac{\partial}{\partial x}g \quad \text{and} \quad \frac{\partial}{\partial x}(kf) = k\frac{\partial}{\partial x}f$$

Similarly for $\frac{\partial}{\partial y}$.

We can also use gradient vector to describe the linear property:

$$\nabla(f+g) = \nabla(f) + \nabla(g) \quad \text{and} \quad \nabla(kf) = k\nabla(f)$$

Theorem. Product Rule:

Let $f(x,y)$ and $g(x,y)$ be two differentiable functions and k a real number. Then we have

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial}{\partial x}g + g\frac{\partial}{\partial x}f$$

Similarly for $\frac{\partial}{\partial y}$.

We can also use gradient vector to describe the linear property:

$$\nabla(fg) = f\nabla(g) + g\nabla(f)$$

Review: The Chain Rule gives the rule for differentiating a composite function $y = f(g(t))$. If we denote the inside function $x = g(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Theorem. Chain Rule

Let $z = f(x,y)$ is a differentiable function. Suppose both $x = g(t)$ and $y = h(t)$ are differentiable functions. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule is also written as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Using the gradient vector,

$$\frac{dz}{dt} = \nabla f \cdot \frac{d}{dt}\langle x, y\rangle$$

Proof.

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Take limit as $\Delta t \to 0$, we obtained.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1. Let $z = f(x, y) = xy^2 + y$, where $x = \sin t$ and $y = \ln t$. Find $\dfrac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= y^2 \cdot \cos t + (2xy+1)\left(\frac{1}{t}\right)$$

$$= (\ln t)^2 \cos t + (2\sin t(\ln t)+1)\left(\frac{1}{t}\right)$$

Example 2. Let $z = f(x, y) = xye^y$, where $x = \sin(t^3)$ and $y = t^2$. Find $\dfrac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= ye^y(\cdot 3t^2\cos(t^3)) + (xe^y + xye^y)2t$$

$$= t^2 e^{t^2}(3t^2 \cos t^3) + \left(\sin t^3 (e^{t^2}) + (\sin t^3) t^2 e^{t^2}\right) 2t$$

Theorem.

Let $z = f(x, y)$ is a differentiable function. Suppose both $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 3. Let $z = f(x, y) = x^2 - y^3$, where $x = st^3$ and $y = s + t^2$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= 2x(t^3) + (-3y^2)1$$
$$= 2st^3t^3 - 3(s+t^2)^2$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= 2x(3st^2) + (-3y^2)2t$$
$$= 2st^3(3st^2) + (-3)(s+t^2)^2 2t$$

Example 4. Let $z = f(x, y) = y \ln x$, where $x = r + t^3$ and $y = rt$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$
$$= \left(\frac{y}{x}\right)1 + (\ln x)t$$
$$= \frac{rt}{r+t^3} + (\ln(r+t^3))\cdot(t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= \frac{y}{x}(3t^2) + (\ln x)(r)$$
$$= \frac{rt}{r+t^3}(3t^2) + (\ln(r+t^3))r$$

Theorem.

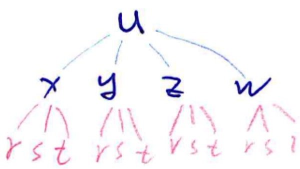
Let $u = f(x, y, z)$ is a differentiable function. Suppose both x, y and z are differentiable functions with variable t. Then u is a differentiable function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Let $u = f(x, y, z, w)$ is a differentiable function. Suppose both x, y, z and w are differentiable functions on variables r, s, t. Then,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial w}\frac{\partial y}{\partial s}$$

The rest $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial t}$ are similar. It is better to use a tree diagram to look at the formula:



Example 5. Let $u = f(x, y, z) = y \ln(xz)$, where $x = rs + t^3$, $y = e^{st}$ and $z = rs^2 t$. Find $\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(\frac{y}{xz}\cdot z\right)(s) + \ln(xz)(0) + \left(\frac{y}{xz}\cdot x\right)\cdot(s^2 t)$$

$$= \frac{e^{st}}{rs+t^3} s + \frac{e^{st}}{rs^2 t}(s^2 t)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left(\frac{y}{x}\right) r + \ln(xz)\left(te^{st}\right) + \left(\frac{y}{z}\right) 2rst$$

$$= \left(\frac{e^{st}}{rs+t^3}\right) r + \left(\ln\left((rs+t^3) rs^2 t\right)\right) te^{st} + \left(\frac{e^{st}}{rs^2 t}\right) 2rst$$

Implicit Differentiation.

Suppose that an equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. That is a function $y = f(x)$ such that $F(x, f(x)) = 0$. If F is differentiable, apply chain rule to $F(x, y) = 0$ respect to x, then

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

We obtained

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

Example 6. Find $\frac{dy}{dx}$ for $x^2 + y^3 + \sin(xy) = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x + y\cos(xy)}{3y^2 + x\cos(xy)}$$

More generally, suppose that an equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y. That is a function $z = f(x, y)$ such that $F(x, y, f(x, y)) = 0$. If F is differentiable, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 7. Find $\frac{\partial z}{\partial x}$ for the implicit function $x^2 + y^2 + z^2 + xyz = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x + yz}{2z + xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y + xz}{2z + xy}$$

Solution for **HW 32:**

The magnitude of net force is $F(x, y, z)$ with position function $\langle x, y, z\rangle = \vec{p}(t) = \langle t^2, t^3 + 2, t + 1\rangle$. When $t = 1$, the position is $\vec{p}(1) = \langle 1, 3, 2\rangle$
The derivative (rate of change) of the magnitude of the net force is

$$\frac{F(t)}{dt} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = F_x(2t) + F_y(3t^2) + F_z$$

The derivative (rate of change) of the magnitude of the net force at $t = 1$ is

$$\frac{F(t)}{dt} = 4.5(2) + 10(3) + (-2) = 37\ N/s$$