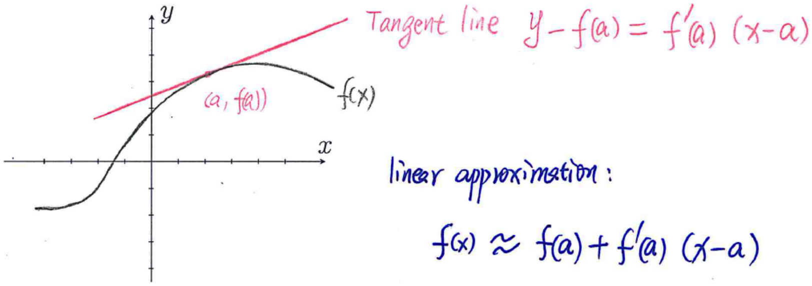
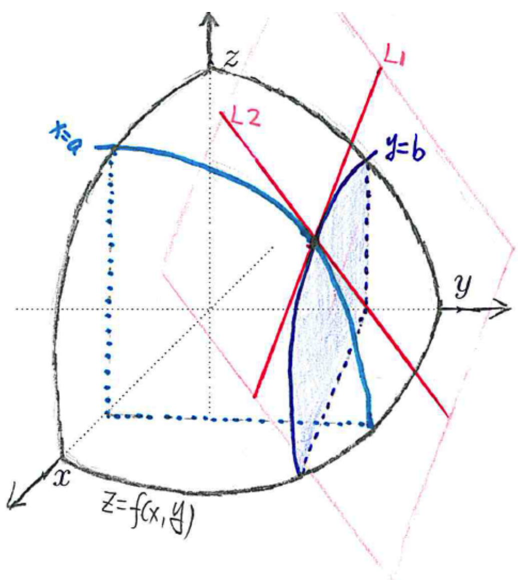
§2.3 Linear approximation, tangent planes, and the differential

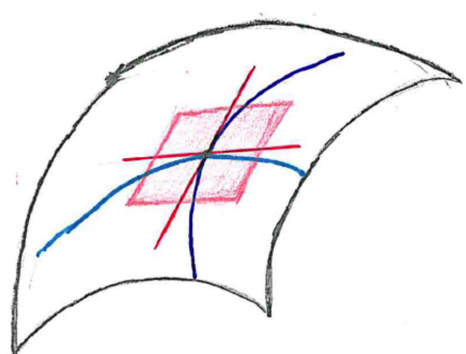
Review: Tangent Lines and Linear Approximations $y = f(x)$ at $x = a$.

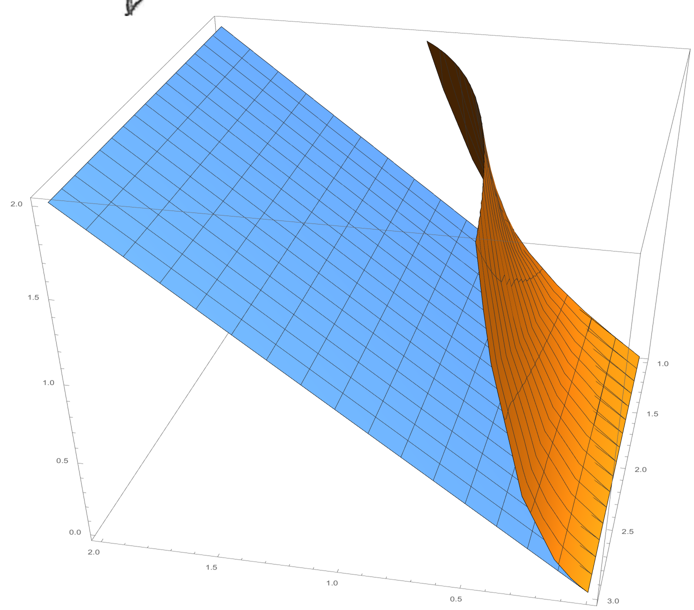


Definition.

Suppose a surface S has equation $z = f(x, y)$. The two cross-sections (traces) of f in the planes $y = b$ and $x = a$ have tangent lines L_1 and L_2 at the point (a, b).
Then the **tangent plane** to the surface S at the point (a, b) is defined to be the plane that contains both tangent lines.







Theorem.

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ is

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Here $c = f(a, b)$.

An compact way to look at the equation of tangent plane is

$$z - f(a, b) = \nabla f(a, b) \cdot \langle x - a, y - b\rangle.$$

If we denote $F(x, y, z) = f(x, y) - z$, the an even compact way to write equation of tangent plane is

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$$

Proof.

① Trace C_1 on $y=b$ has equation $z=f(x,b)$.

Tangent line L_1 has slope $f_x(a,b)$, tangent vector $\vec{v_1} = \langle 1, 0, f_x(a,b)\rangle$

② Trace C_2 on $x=a$ has equation $z=f(a,y)$.

Tangent line L_2 has slope $f_y(a,b)$, tangent vector $\vec{v_2} = \langle 0, 1, f_y(a,b)\rangle$

③ Normal vector of the tangent plane:

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = -f_x(a,b)\,\vec{i} - f_y(a,b)\,\vec{j} + \vec{k}$$

$$= \langle -f_x(a,b), -f_y(a,b), 1\rangle$$

④ The equation for tangent plane:

$$\langle x-a, y-b, z-c\rangle \cdot \langle -f_x(a,b), -f_y(a,b), 1\rangle = 0$$

or $z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$. ✓

Example 1. Find the tangent plane to the elliptic paraboloid $z = 3x^2 + y^2$ at the point $(1, 2, 7)$.

$f(x, y) = 3x^2 + y^2$.
$f_x(x, y) = 6x$ gives $f_x(1, 2) = 6$.
$f_y(x, y) = 2y$ gives $f_y(1, 2) = 4$.
The tangent plane has an equation $z - 7 = 6(x - 1) + 4(y - 2)$, or $z = 6x + 4y - 7$

The function $L(x, y)$ for the above tangent plane is called the **linearization** of f at $P = (1, 2)$ and the approximation

$$f(x, y) \approx 6x + 4y - 7$$

is called **the linear approximation** or **tangent plane approximation** of f at $P = (1, 2)$.

We can use the linear approximation to approx the value of the function near the point P.

For instance, at the point $Q = (1.1, 1.9)$, the linear approximation gives $f(1.1, 1.9) \approx 7.2$.

The precise value $f(1.1, 1.9) = 3(1.1^2) + (1.9^2) = 7.24$.

Definition.

The **linear approximation** $f(x, y)$ at the point (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the data $f_x(a, b)$ and $f_y(a, b)$ is captured by the gradient vector

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

The linear approximation formula can be written as **Differential Approximation**

$$\Delta(z) \approx \nabla f(a, b) \cdot \Delta(x, y).$$

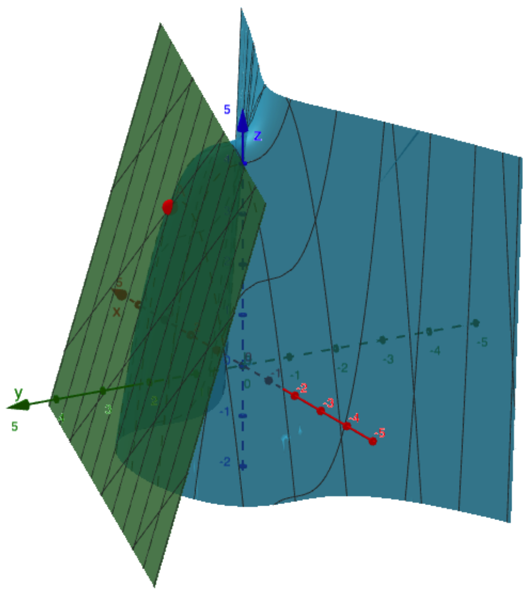
Here $\Delta(z)$ is the difference of the value $f(x, y) - f(a, b)$, and $\Delta(x, y)$ is the difference vector $\langle x, y\rangle - \langle a, b\rangle$.

Example 2. Suppose $f(2, 1) = 13$, $\nabla f(2, 1) = \langle 12, 2\rangle$. Estimate the value of f at $Q = (2.1, 0.9)$.

$f(2.1, 0.9) - f(2, 1) \approx \nabla f(2, 1) \cdot \Delta(x, y) = \langle 12, 2\rangle \cdot \langle 0.1, -0.1\rangle$
So, $f(2.1, 0.9) \approx 14$.
The precise is $f(2.1, 0.9) = 14.04$.

Example 3. Find the tangent plane to the elliptic paraboloid $z = 4 + x - x^2 - y^3$ at the point $(1, 1)$. Look at the graph at https://www.geogebra.org/3d

The Tangent Plane is $z - 3 = -(x - 1) - 3(y - 1)$ or $L(x, y) = 3 - (x - 1) - 3(y - 1)$.
The graph is



Review: Differentialble function $z = f(x)$ at $x = a$.

$\Delta z = f'(a)\, \Delta x + \varepsilon \Delta x$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

$f(x) - f(a) \approx f'(a)(x - a)$ good approximation.

Definition.

The function $z = f(x, y)$ is **differentiable** at (a, b) if Δz can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

A differentiable function is one for which the linear approximation is a **good** approximation when (x, y) is near (a, b).

Theorem.

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 4. Show that $f(x,y) = xe^{xy} - x^2$ is differentiable at $(0,1)$ and find its linearization there.

$f_x(x,y) = e^{xy} + xye^{xy} - 2x$

$f_y(x,y) = x^2 e^{xy}$

Both $f_x(x,y)$ and $f_y(x,y)$ exist and continuous in $\mathbb{R}^2$.

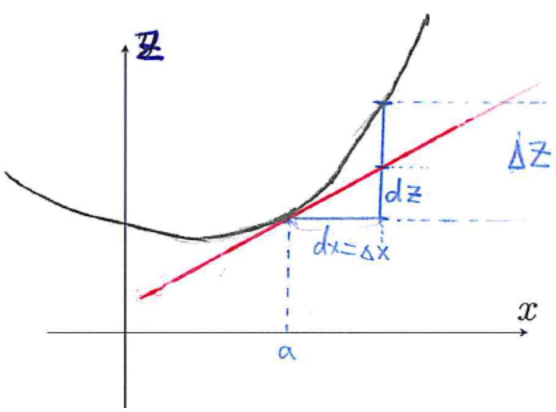
So $f(x,y)$ is differentiable at $(0,1)$

$$L(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)$$
$$= 0 + (x-0) + 0$$
$$= x$$

Definition.

For a differentiable function $z = f(x)$, the **differentials** dx is an independent variable, (i.e., it can be given any values). The **differential** dz is defined by

$$dz := f'(x)dx = \frac{dz}{dx}dx$$



$\Delta z \approx dz$

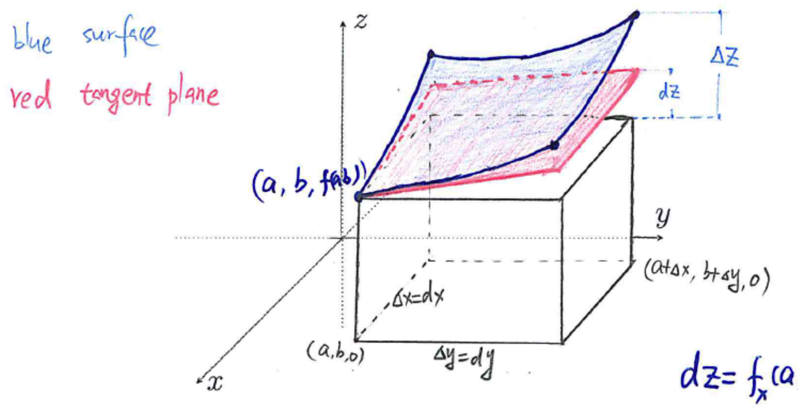
$f(x) \approx f(a) + f'(a)(x-a)$

$f(x) \approx f(a) + dz$

Definition.

For a differentiable function $z = f(x,y)$ of two variables, we define the **differentials** dx and dy to be independent variables, (i.e., they can be given any values). The **differential** dz is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$



$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(x,y) \approx f(a,b) + dz$$

Example 5. (a) If $z = f(x, y) = x^3 + 2xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.03 and y changes from 3 to 2.95, compare the values of Δz and dz.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (3x^2+2y)dx + (2x-2y)dy$$

$x=2$
$y=3$
$dx=0.03$
$dy=0.05$

$$dz = (3(2)^2+2(3))\,0.03 + (2(2)-2(3))(-0.05)$$
$$= 0.64$$

$$\Delta z = f(2.03, 2.95) - f(2,3)$$
$$= 11.639927 - 11$$
$$= 0.639927$$

Example 6. The length and width of a **rectangle** are measured as 52 cm and 50 cm, respectively, with an error in measurement of at most 0.02 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

Area = length × width

$$A = xy$$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$
$$= y\,dx + x\,dy$$

Information

$x = 52$ cm $y = 50$ cm

$\Delta x \leq 0.02$ cm

$\Delta y \leq 0.02$ cm

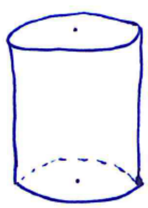
Maximum Error $= \Delta A \approx dA$

$$dA = 50(0.02) + 52(0.02)$$
$$= 2.04 \text{ cm.}$$

HW22. A cardboard box is measured to have length, width, and height of 2, 3, and 1 feet, respectively, to enclose a volume of 6 cubic feet. However, more-careful measurements show that the box is really 2.01 by 2.98 by 1.03 feet. (a) Use linear approximation to estimate the revised (measured) volume of the box. (b) Use differential approximation to estimate the change in the (measured) volume of the box. (c) Show that your answers to part (a) and (b) agree.

Let x, y, z be the length, width, and height respectively.
The volume is $V = xyz$.
(a) $V(x,y,z) \approx V(a,b,c) + V_x(a,b,c)(x-a) + V_y(a,b,c)(y-b) + V_z(a,b,c)(z-c)$
So $V(2.01, 2.98, 1.03) \approx V(2,3,1) + 4(0.01) + 2(-0.02) + 6(0.03) = 6.18$
(b) $dV = V_x dx + V_y dy + V_z dz = yzdx + xzdy + xydz$
$dV(2,3,1) = 4(0.01) + 2(-0.02) + 6(0.03) = 0.18$
(c). $dV(2,3,1) \approx \Delta V = 6.169 - 6$

Example 7. Use differentials to estimate the amount of metal in a closed **cylindrical** can that is 26 cm high and 8 cm in diameter if the metal in the top and the bottom is 0.3 cm thick and the metal in the sides is 0.06 cm thick.



diameter $= 8 \Longrightarrow$ radius $r = 4$ cm

Volume $V = \pi r^2 h$

$$\Delta V \approx dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = 2\pi rh\,dr + \pi r^2 dh$$

$dr = 0.06$

$dh = 2 \times 0.3 = 0.6$

$$dV = 2\pi(4)(26)(0.06) + \pi(4^2)\,0.6$$
$$= 22.08\pi \approx 69.33 \text{ cm}^3$$

Example 8. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation

$$PV = 8.25T$$

where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 11 L to 11.2 L and the temperature decreases from 310 K to 300 K.

$\Delta V = 0.2 L$ $\Delta T = -10\ K$

$P = 8.25\, T\, (V^{-1})$

$$\Delta P \approx dP = \frac{\partial P}{\partial T} dT + \frac{\partial P}{\partial V} dV$$

$$= 8.25 (V^{-1})\, dT + (-8.25)\, T (V^{-2})$$

$$= 8.25 \left(\frac{1}{11}\right)(-10) - 8.25\, (310) \left(\frac{1}{11^2}\right)$$

$$= 8.25 \left(\frac{-420}{121}\right)$$

$$\approx -28.63636 \quad k\cdot Pa$$

$V = 11$
$T = 310$