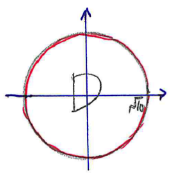
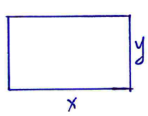
§2.11 Lagrange Multipliers

Example (a). Find the extreme values of $f(x, y) = 2x + 6y$ on the boundary of the region D described by the equation $x^2 + y^2 = 10$.

This example is the Step 2 of finding absolute maximum or minimum.



Example (b). A rectangular made from 12 m (meters) of line. Find the maximum area of such a rectangle.

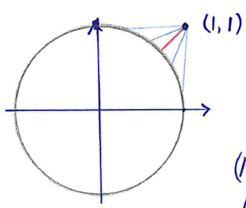


- Area $= xy$ • perimeter $= 2x+2y=12$

$f(x,y)=xy$ $\quad x+y=6$

Maximize $f(x,y)=xy$ when $x+y=6$.

Example (c). Find the shortest distance from the point $(1, 1)$ to the circle $x^2 + y^2 = 1$.

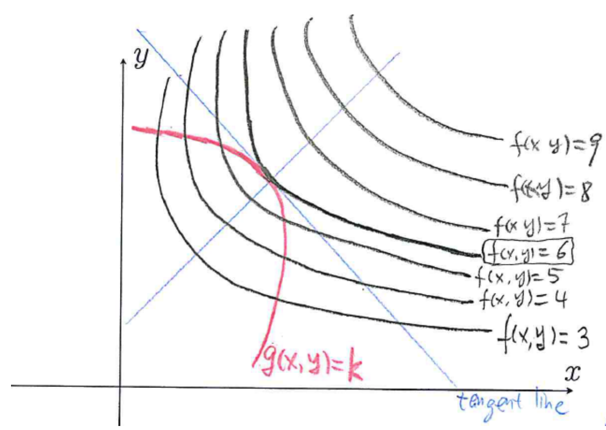


Distance from $(1,1)$ to (x,y) is

$f(x,y)=\sqrt{(x-1)^2+(y-1)^2}$

(Maximize)
Minimize $f(x,y)$ on the circle $\overset{\text{Domain}}{x^2+y^2=1}$.

Example (a,b,c). Find extreme value of a function $z = f(x, y)$ subject to a constraint $g(x, y) = k$. Here, k is a given number.



Draw the level curves for the functions $z = f(x, y)$:

The Extreme Value is $f(x, y) = 6$.

$g(x, y) = k$ and $f(x, y) = 6$ have the same tangent line.

Their normal vectors ∇f and ∇g have the same direction.

Motivation question: Find extreme value of a function $z = f(x, y)$ subject to a constraint $g(x, y) = k$. Here, k is a given number.

Theorem. Method of Lagrange Multipliers:

(1) Find all values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

and

$$g(x, y) = k$$

(2) Evaluate f at all points (x, y) from step (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.
Equations in (1) can be written as

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$$

There is a technical assumption for Lagrange Multiplier: for all (x, y) such that $g(x, y) = k$, we need $\nabla g(x, y) \neq \vec{0}$.

Example 1. Find the extreme values of $f(x, y) = 2x + 6y$ on the boundary of the region D described by the equation $x^2 + y^2 = 10$.

Step 1: $\nabla f(x,y) = \langle 2, 6 \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2 = 2\lambda x \\ 6 = 2\lambda y \end{cases} \Rightarrow \begin{cases} \lambda x = 1 \\ \lambda y = 3 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\lambda} \\ y = \frac{3}{\lambda} \end{cases}$

together with $x^2+y^2=10$

$\Rightarrow \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 10 \Rightarrow \frac{10}{\lambda^2} = 10 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

Step 2: Possible extreme points are $(1, 3)$, $(-1, -3)$

Compute $f(1, 3) = 20$

$f(-1, -3) = -20$

So $f(x, y) = 2x + 6y$ on constraint $x^2+y^2=10$

has maximum value $f(1,3) = 20$

minimum value $f(-1,-3) = -20$

Example 2. Find the extreme values of $f(x, y) = x^2+2y^2+2$ on the circle equation $x^2+y^2 = 1$.

Step 1: $\nabla f(x,y) = \langle 2x, 4y \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

Lagrange multiplier:
$$\begin{cases} 2x = 2\lambda x & \Rightarrow x(\lambda-1)=0 \Rightarrow x=0 \text{ or } \lambda=1 \\ 4y = 2\lambda y & \Rightarrow y(\lambda-2)=0 \Rightarrow y=0 \text{ or } \lambda=2 \\ x^2+y^2=1 \end{cases}$$

- If $x=0$ then $y=\pm 1$. $\Rightarrow$ possible extreme points $(0,1)$ and $(0,-1)$
- If $\lambda=1$, then $y=0$, then $x=\pm 1$.

$\Rightarrow$ possible extreme points $(1,0)$, $(-1,0)$.

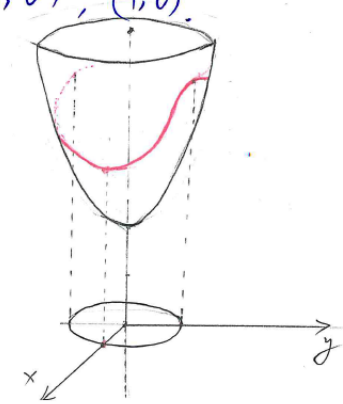
Step 2. $f(0,1)=4$

$f(0,-1)=4$

$f(1,0)=3$

$f(-1,0)=3$

So, $f(x,y)$ on circle has minimum 3 and maximum 4.



Remark: there is a easier way to find the result for this particular example.

Motivation question: Find extreme value of a function $u = f(x, y, z)$ subject to a constraint $g(x, y, z) = k$. Here, k is a given number.

Theorem. Method of Lagrange Multipliers:

(1) Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(2) Evaluate f at all points (x, y, z) from step (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.
Equations in (1) can be written as

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x, y, z) = k \end{cases}$$

Example 3. A rectangular box is to be made from 24 m^2 of cardboard. Find the maximum volume of such a box.

The volume is $V = f(x, y, z) = xyz$. The constraint $2xy + 2yz + 2xz = 24$, or $g(x, y, z) = xy + yz + xz = 12$.

Step 1: $\nabla f = \langle yz, xz, xy \rangle$ $\quad \nabla g = \langle y+z, x+z, x+y \rangle$

Lagrange multipliers: $\begin{cases} yz = \lambda(y+z) \\ xz = \lambda(x+z) \\ xy = \lambda(x+y) \\ xy + yz + xz = 12 \end{cases} \Rightarrow \begin{cases} xyz = \lambda(xy + xz) \\ xyz = \lambda(xy + yz) \\ xyz = \lambda(xz + yz) \\ xy + yz + xz = 12 \end{cases}$

observe: $\lambda \neq 0$.

- The first two equations give $xz = yz \Rightarrow x = y$
- The first and third equations give $xy = yz \Rightarrow x = z$
- The second and third equations give $xy = xz \Rightarrow y = z$

plug in to constraint $g(x,y,z) = 12 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = 2$.

So the only possible point is $(2, 2, 2)$.

The Maximum volume is $V = f(2, 2, 2) = 8m^3$.

Example 4. (Compare with example (c)) Find the shortest distance from the point $(1, 2, 3)$ to the sphere

$$x^2 + y^2 + z^2 = 4$$

· Distance function $d = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$

· we only find extreme value on $d^2 = (x-1)^2+(y-2)^2+(z-3)^2$.

So. Let $f(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2$ with constraint $g(x,y,z)=x^2+y^2+z^2=4$.

Step 1: $\nabla f = \langle 2(x-1),\ 2(y-2),\ 2(z-3)\rangle$ $\quad \nabla g = \langle 2x, 2y, 2z\rangle$

Lagrange Multipliers: $\begin{cases} 2(x-1)=2\lambda x & \Rightarrow x = \frac{1}{1-\lambda} \\ 2(y-2)=2\lambda y & \Rightarrow y=\frac{2}{1-\lambda} \\ 2(z-3)=2\lambda z & \Rightarrow z=\frac{3}{1-\lambda} \\ x^2+y^2+z^2=4 \end{cases}$

$$\Rightarrow \left(\frac{1}{1-\lambda}\right)^2+\left(\frac{2}{1-\lambda}\right)^2+\left(\frac{3}{1-\lambda}\right)^2=4 \quad \Rightarrow (1-\lambda)^2=\frac{14}{4} \quad \Rightarrow 1-\lambda=\pm\frac{\sqrt{14}}{2}$$

$$\Rightarrow \lambda = 1\pm\frac{\sqrt{14}}{2} \quad \Rightarrow \begin{cases} x=\frac{2}{\sqrt{14}} \\ y=\frac{4}{\sqrt{14}} \\ z=\frac{6}{\sqrt{14}} \end{cases} \text{ and } \begin{cases} x=-\frac{2}{\sqrt{14}} \\ y=-\frac{4}{\sqrt{14}} \\ z=-\frac{6}{\sqrt{14}} \end{cases}$$

closest point to $(1,2,3)$ farthest point to $(1,2,3)$

Step 2: Evaluate these two points on $d=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$.

Motivation Question. Find extreme values of a function $u = f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. Here, k and c is are given numbers.

(Temperature example)

Theorem. Method of Lagrange Multipliers:

(1) Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

and

$$g(x, y, z) = k$$

and

$$h(x, y, z) = c$$

(2) Evaluate f at all points (x, y, z) from step (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Equations in (1) can be written as

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x, y, z) = k \\ h(x, y, z) = c \end{cases}$$

Example 5. Find the extreme values of $f(x, y, z) = xy + yz$ subject to two constraints, $xy = 1$, and $y^2 + z^2 = 9$.

Step 1:

$\nabla f = \langle y, x+z, y \rangle \quad \nabla g = \langle y, x, 0 \rangle \quad \nabla h = \langle 0, 2y, 2z \rangle$

Lagrange multiplier $\nabla f = \lambda \nabla g + \mu \nabla h$

$$\begin{cases} y = \lambda y \implies \lambda = 1 \quad \text{since } y \neq 0 \\ x+z = \lambda x + 2\mu y \\ y = 2\mu z \\ xy = 1 \\ y^2+z^2 = 9 \end{cases} \Rightarrow \begin{cases} x+z = x+2\mu y \implies z = 2\mu y \\ y = 2\mu z \\ xy = 1 \\ y^2+z^2=9 \end{cases}$$

$$\Rightarrow \begin{cases} 2\mu = \frac{z}{y} \\ 2\mu = \frac{y}{z} \end{cases} \Rightarrow \frac{z}{y} = \frac{y}{z} \Rightarrow y^2 = z^2$$

$$\begin{cases} y^2 = z^2 \\ y^2+z^2 = 9 \end{cases} \Rightarrow 2z^2 = 9 \quad z = \pm\frac{3}{\sqrt{2}}$$

$$\Rightarrow y = \pm\frac{3}{\sqrt{2}}$$

$$xy = 1 \Rightarrow x = \frac{1}{y} \Rightarrow x = \pm\frac{\sqrt{2}}{3}$$

All possible points are $\left(\pm\frac{\sqrt{2}}{3}, \pm\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ and $\left(\pm\frac{\sqrt{2}}{3}, \pm\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$

Plug in all points $f\left(\pm\frac{\sqrt{2}}{3}, \pm\frac{3}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\right) = 1 + \frac{9}{2} = \frac{11}{2}$

$$f\left(\pm\frac{\sqrt{2}}{3}, \pm\frac{3}{\sqrt{2}}, \mp\frac{3}{\sqrt{2}}\right) = 1 - \frac{9}{2} = -\frac{7}{2}$$