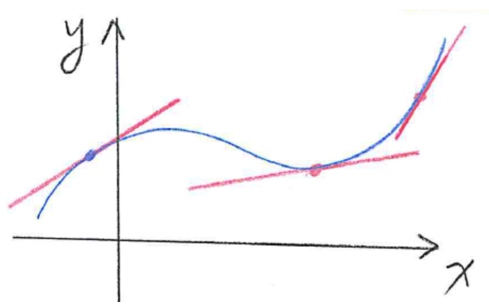
§2.1 Partial Derivatives

Review: The derivative of $f(x)$ is

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Geometrically, $f'(x)$ is the slope of the tangent line.



Definition.

The **partial derivative of** $f(x,y)$ **with respect to** x **at** (a,b) is

$$f_x(a,b) = g'(a),$$

where $g(x) = f(x,b)$ is the trace of $f(x,y)$ in plane $y=b$.
That is

$$f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

The **partial derivative of** $f(x,y)$ **with respect to** y **at** (a,b) is

$$f_y(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

Definition.

The **partial derivative of** $f(x,y)$ **with respect to** x **(and to** y**)**

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Notations: If $z = f(x,y)$,

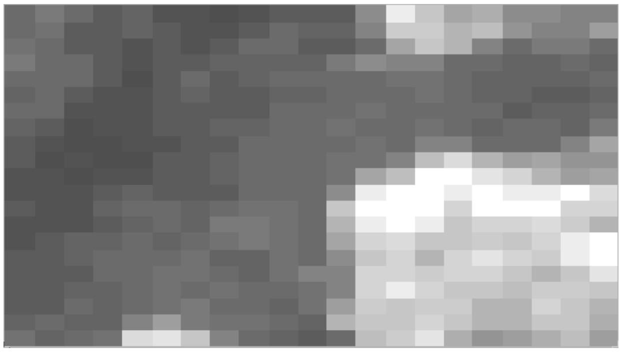
$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} = D_x f = f_1 = D_1 f$$

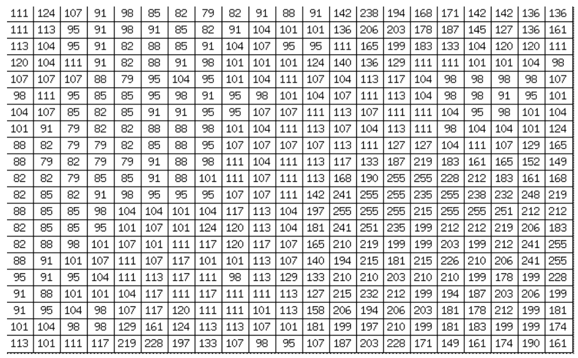
Remark: Definition of partial derivative can be used as estimating the partial derivative.

For example, suppose we know $f(1,2)$, $f(1,2.15)$ and $f(1.1,2)$, then we can estimate the partial derivative

$$f_x(1,2) \approx \frac{f(1.1,2)-f(1,2)}{0.1} \quad \text{and} \quad f_y(1,2) \approx \frac{f(1,2.15)-f(1,2)}{0.15}$$

Application: This is useful in image processing(edge detection) for a matrix of given data.





111	124	107	91	98	85	82	79	82	91	88	91	142	238	194	168	171	142	142	136	136
111	113	95	91	98	91	85	82	91	104	101	101	136	206	203	178	187	145	127	136	161
113	104	95	91	82	88	85	91	104	107	95	95	111	165	199	183	133	104	120	120	111
120	104	111	91	82	88	91	98	101	101	101	124	140	136	129	111	111	101	101	104	98
107	107	107	88	79	95	104	95	101	104	111	107	104	113	117	104	98	98	98	98	107
98	111	95	85	85	95	98	91	95	98	101	104	107	111	113	104	98	98	91	95	101
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88	79	82	79	79	91	88	98	111	104	111	113	117	133	187	219	183	161	165	152	149
82	82	79	85	85	91	88	101	111	107	111	113	168	190	255	255	228	212	183	161	168
82	85	82	91	98	95	95	95	107	107	111	142	241	255	255	235	255	238	232	248	219
88	85	85	98	104	104	101	104	117	113	104	197	255	255	255	215	255	255	251	212	212
82	85	85	95	101	107	101	124	120	113	104	181	241	251	235	199	212	212	219	206	183
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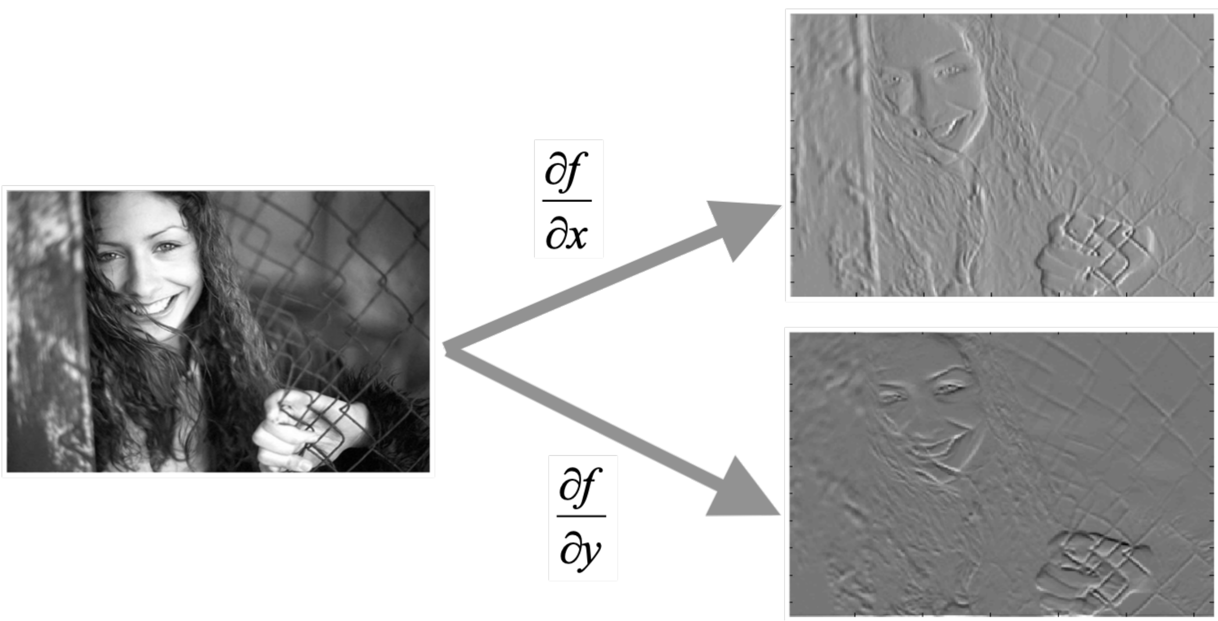


Rule for finding partial derivative:
To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
Similarly for f_y.

Example 1. If $f(x,y) = x^3 + x^2 \sin y - xe^y$, find $f_x(1,2)$ and $f_y(1,2)$.

$f_x = 3x^2 + 2x\sin y - e^y$ $\qquad$ $f_x(1,2) = 3 + 2\sin 2 - e^2$

$f_y = x^2\cos y - xe^y$ $\qquad$ $f_y(1,2) = \cos 2 - e^2$

Example 2. If $f(x,y) = \sqrt{x^2+y^2}$, find $f_x(x,y)$ and $f_y(x,y)$, $f_x(1,2)$.

Rewrite $f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$.

$f_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$ Chain Rule

$f_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

$f_x(1,2) = 1/\sqrt{5}$.

Example 3. If $z = f(x,y)$ is defined implicitly as

$$x^3 + y^3 + z^3 + 2xyz = 3,$$

find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Take $\frac{\partial}{\partial x}$ of the equation

$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 2yz + 2xy \frac{\partial z}{\partial x} = 0$

$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{3x^2+2yz}{3z^2+2xy}$

Take $\frac{\partial}{\partial y}$ of the equation

$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 2xz + 2xy \frac{\partial z}{\partial y} = 0$

$\Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{3y^2+2xz}{3z^2+2xy}$

Example 4. If $f(x,y) = \arccos x + x\sin^2(3x+y)$, find $f_x(0,0)$ $f_x(x,y)$ and $f_y(x,y)$.

$f_x(x,y) = -\frac{1}{\sqrt{1-x^2}} + \sin^2(x+y) + 6x\sin(3x+y)\cos(3x+y)$

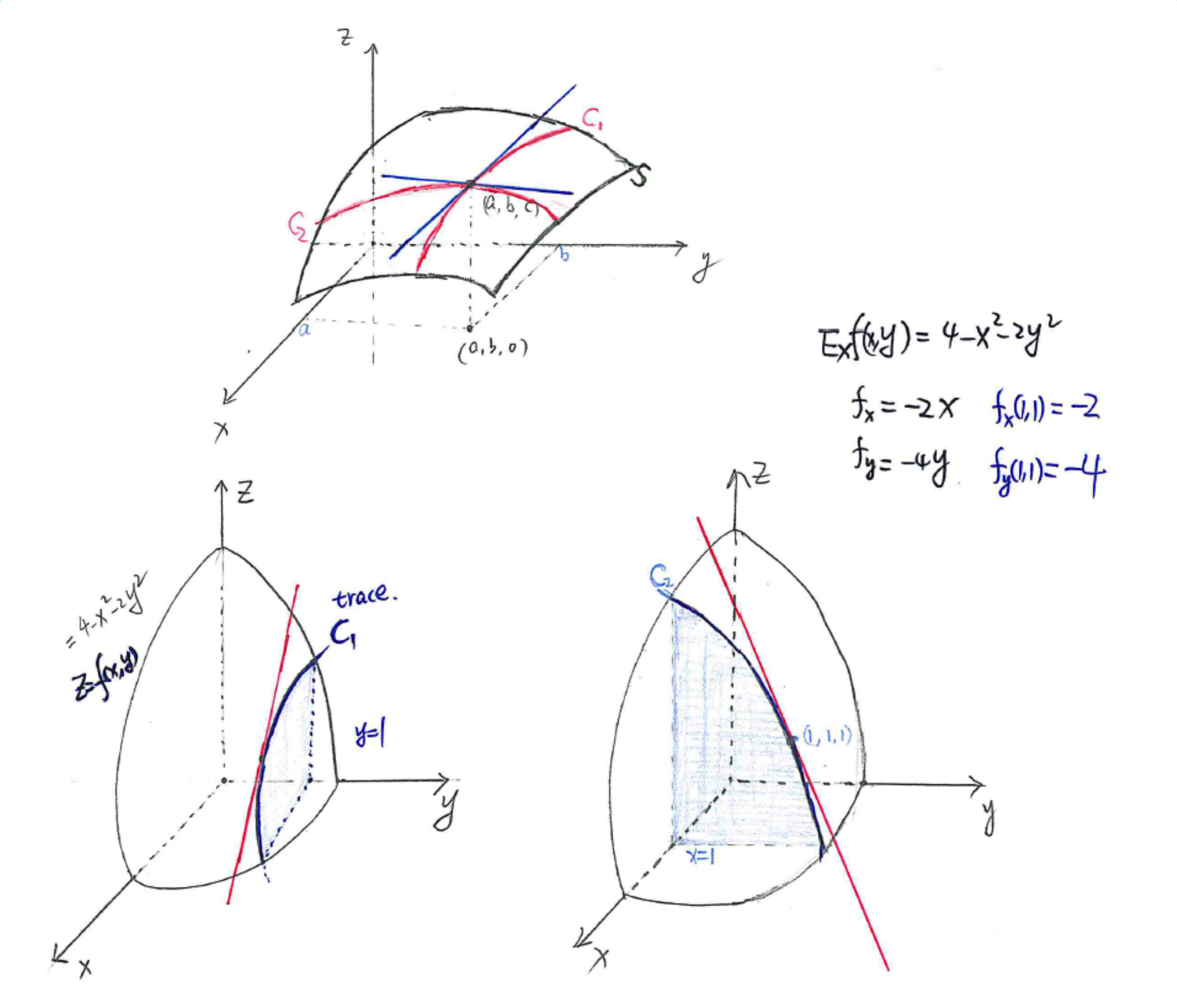
$f_y(x,y) = 2x\sin(3x+y)\cos(3x+y)$

$f_x(0,0) = -1$.

Geometric interpretation of partial derivatives

The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ are derivative of the traces of $z = f(x,y)$ in the planes $y = b$ and $x = a$.

The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the ***slopes of the tangent lines*** at (a,b) to the traces(cross-sections) C_1 ($z = f(x,b)$) and C_2 ($z = f(a,y)$) of S in the planes $y = b$ and $x = a$.



Second partial derivatives

The **second partial derivatives** of $z = f(x,y)$ are denoted by

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 5. Find the second partial derivative f_{xy} of $f(x, y) = x^2 + xy^3 + \sin(xy)$.

$f_x = 2x + y^3 + \cos(xy) \cdot y$ $\quad f_y = 3xy + \cos(xy)x$

$f_{xx} = 2 - \sin(xy)y^2$

$f_{xy} = 3y^2 - \sin(xy)xy + \cos xy$

$f_{yx} = 3y^2 - \sin(xy)xy + \cos xy$

$f_{yy} = 6xy - \sin(xy)x^2$

Theorem. (Clairaut's Theorem)

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

The Gradient Vector

Definition.

The **gradient** of a function $f(x, y)$ is the vector function ∇f by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

The notation ∇ is pronounced Del or nabla.

The formula for the directional derivative in the direction of $\vec{u} = \langle a, b\rangle$

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

or $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ for short.

Example 6. (1) Find the gradient of $f(x, y) = \sin(xy) + e^y$ at $(0, 1)$.

$$\nabla f(x, y) = \langle y\cos(xy), x\cos(xy) + e^y\rangle$$
$$\nabla f(0, 1) = \langle 1, e\rangle$$

(2) Find the directional derivative of $f(x, y)$ at $(0, 1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 3\vec{j}$.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{4+9}}\langle 2, 3\rangle = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle$$
$$D_{\vec{u}}f = \langle 1, 3\rangle \cdot \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle = \frac{2}{\sqrt{13}} + \frac{3e}{\sqrt{13}} = \frac{2+3e}{\sqrt{13}}$$