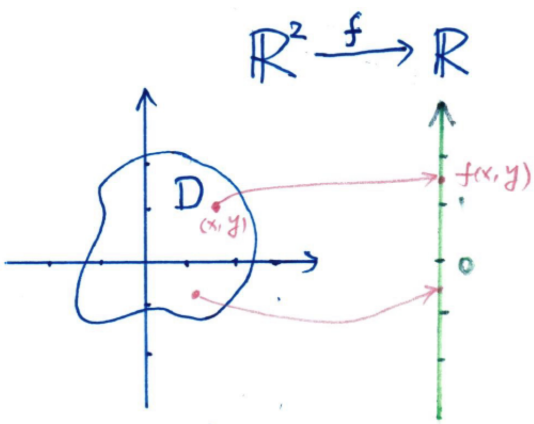
§2.0 Multi-variable functions and Limits

1. Functions of several variables

Definition.

A **function of two variables** is a rule that assigns to each ordered pair $(x, y) \in D \subset \mathbb{R}^2$ of real numbers a unique $f(x, y) \in \mathbb{R}$. The set D is the **domain** of and its **range** is the set $\{f(x, y)|(x, y) \in D\}$.



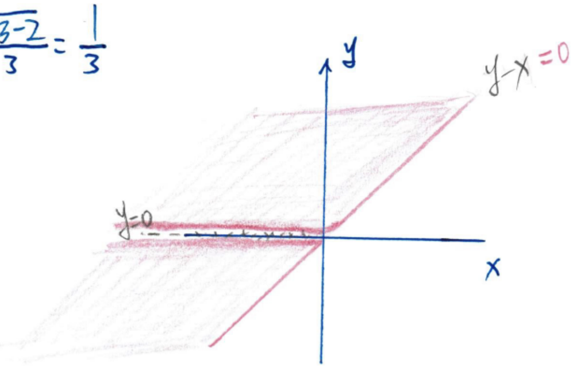
Example 1. The temperature T at a point (x, y) on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

Example 2. Let $f(x, y) = \dfrac{\sqrt{y - x}}{y}$. Evaluate $f(2, 3)$, find and sketch the domain.

$f(2,3) = \dfrac{\sqrt{3-2}}{3} = \dfrac{1}{3}$

$\begin{cases} y - x \geq 0 \\ y \neq 0 \end{cases}$

or $\begin{cases} y \geq x \\ y \neq 0 \end{cases}$



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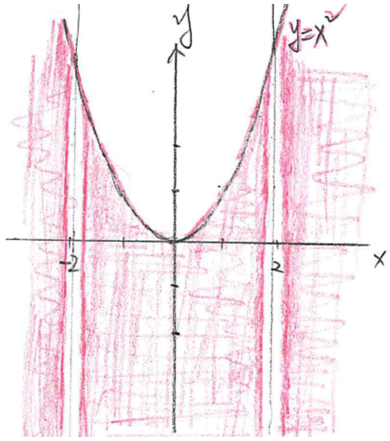
Example 3. Let $f(x, y) = \dfrac{\sqrt{x^2 - y}}{x^2 - 4}$. Evaluate $f(3, 2)$, find and sketch the domain.

$$f(3,2) = \frac{\sqrt{9-2}}{9-4} = \frac{\sqrt{7}}{5}$$

domain $\begin{cases} x^2 - y \geq 0 \\ x^2 - 4 \neq 0 \end{cases}$

or $\begin{cases} y \leq x^2 \\ x \neq \pm 2 \end{cases}$



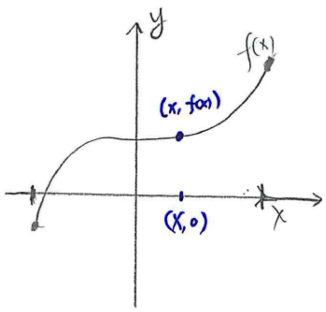
Definition.

The **graph** of $f(x,y)$ is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in D$. That is

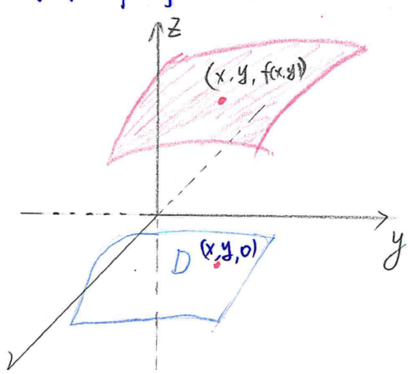
$$\{(x,y,f(x,y)) \in \mathbb{R}^3 | (x,y) \in D\}$$

The graph of $f(x,y)$ is a surface above/below the domain.

graph of f(x)

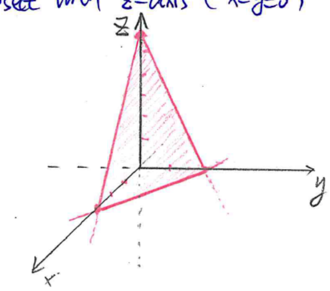


graph of f(x, y)

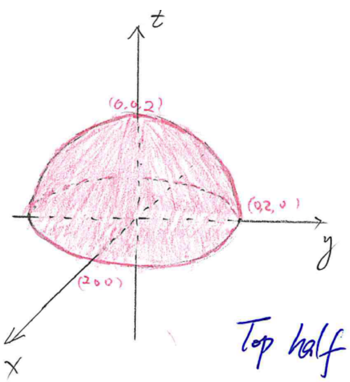


Example 4. Sketch the graph of the function $f(x,y) = 6 - 2x - 3y$.

Intersect with x-axis ($y=z=0$) $(3,0,0)$
Intersect with y-axis ($x=z=0$) $(0,2,0)$
Intersect with z-axis ($x=y=0$) $(0,0,6)$



Example 5. Sketch the graph of the function $f(x,y)=\sqrt{4-x^2-y^2}$.



$z=\sqrt{4-x^2-y^2}$

$z^2=4-x^2-y^2$

$x^2+y^2+z^2=2^2$

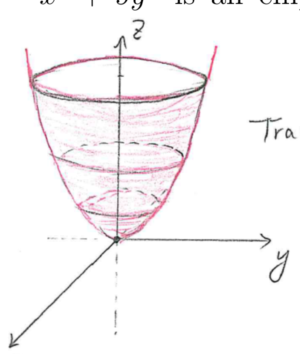
$\underline{z\geq 0}$.

Top half of the sphere.

The domain is $4-x^2-y^2\geq 0$, or equivalently, $x^2+y^2\leq 2^2$. The range is $0\leq z\leq 2$.

Example 6. Find the domain and range, and sketch the graph of the function $f(x,y)=x^2+9y^2$.

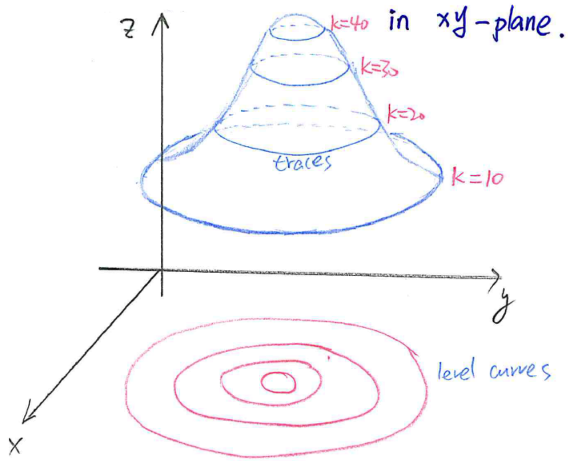
$z=x^2+9y^2$ is an elliptic paraboloid. The domain is $\mathbb{R}^2$ and the range is $z\geq 0$.



Traces : Elliptic

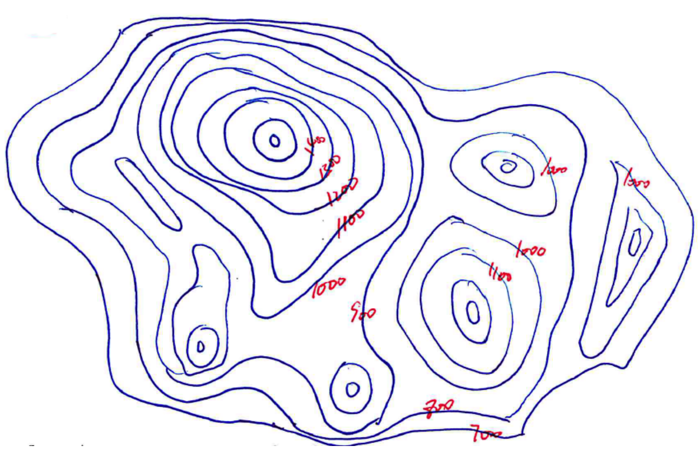
Definition.

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = c$, where is c a constant (in the range of f).

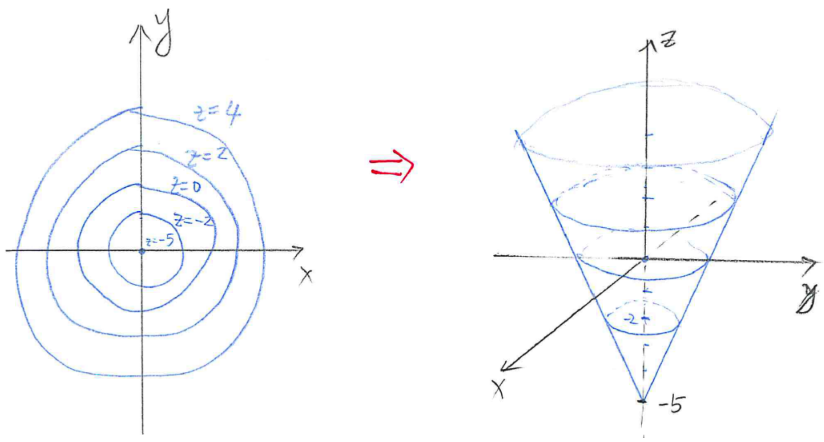


The level curves $f(x, y) = c$ are just the traces of the graph of f in the horizontal plane $z = c$ projected down to the xy-plane.

One common example of level curves (contour map) occurs in topographic maps of mountainous regions.

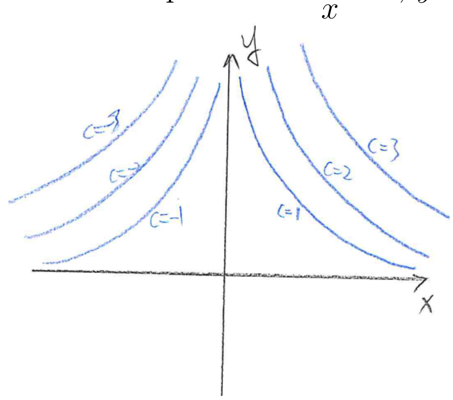


Example 7. A contour map of a function is shown. Use it to make a rough sketch of the graph of f.



Example 8. Draw a contour map of the function showing several level curves. $f(x,y) = xe^{4y}$.

$xe^{4y} = c$ implies $e^{4y} = \dfrac{c}{x}$. So, $y = \dfrac{1}{4}\ln(\dfrac{c}{x})$.



A function of three variables.

$f : \mathbb{R}^3 \to \mathbb{R}$ is denoted by $f(x,y,z)$.

For example, $f(x,y,z) = \dfrac{\sqrt{x^2+y^2+z^2}}{x}$

More generally, one can define a function of multi-variables: $f : \mathbb{R}^n \to \mathbb{R}$, denoted by $f(x_1, x_2, \ldots, x_n)$.

For example, $f(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$

2. Limits and Continuity

Definition.

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

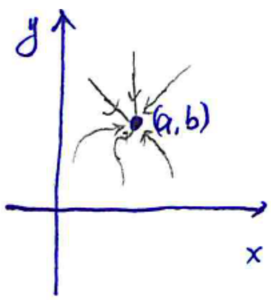
if for every number $\epsilon > 0$, there is a corresponding number $\delta(\epsilon) > 0$ such that if $(x, y) \in D$ and $0 < \text{dist}((x, y), (a, b)) < \delta$ then

$$|f(x, y) - L| < \epsilon.$$

Here, $\text{dist}((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}$.

This is the precise definition for limit of a function, and referred as $\epsilon - \delta$ definition. It is not required for a Calculus class. More details about the precise foundation of calculus is available on a Mathematical Analysis class or Real Analysis class.

From definition, it means that we need to approach (a, b) from any direction.



Example 9. $\lim_{(x,y)\to(0,0)} e^{-xy} \sin(x + y) = 0$

Example 10. $\lim_{(x,y)\to(1,2)} \dfrac{x + y^2}{x^2 - y^2} = \dfrac{5}{-3}$

Definition.

A function f of two variables is called **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Example 11. Compute $\lim_{(x,y)\to(0,0)} f(x, y)$ and $\lim_{(x,y)\to(1,1)} f(x, y)$ for $f(x, y) = \dfrac{x^4 - y^4}{x - y}$

$$f(x,y) = \frac{(x^2-y^2)(x^2+y^2)}{x-y} = \frac{(x-y)(x+y)(x^2+y^2)}{(x-y)} = (x+y)(x^2+y^2)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\lim_{(x,y)\to(1,1)} f(x,y) = 2\times 2 = 4$$

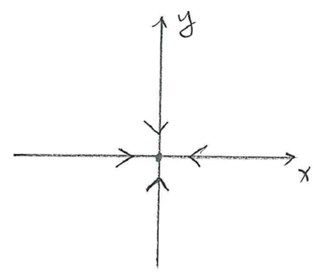
Example 12. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ for $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

- Approach $(0,0)$ along x-axis ($y=0$)

  $f(x,0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$.

- Approach $(0,0)$ along y-axis ($x=0$)

  $f(0,y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$.



f has different limits along two different lines

Then. $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 13. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ for $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

- Approach $(0,0)$ along x-axis $(y=0)$

  $f(x,0) = 0$ for $x \neq 0$.

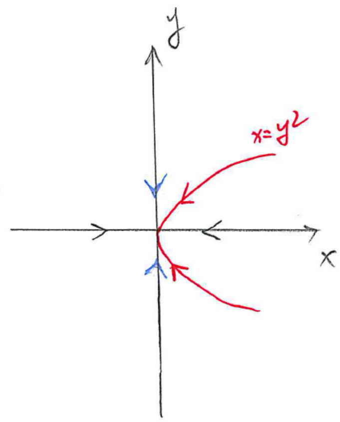
- Approach $(0,0)$ along y-axis $(x=0)$

  $f(0,y) = 0$ for $y \neq 0$.

- Approach $(0,0)$ along $x = y^2$

  $$f(x,y) = \frac{y^2 y^2}{y^4 + y^4} = \frac{1}{2}$$

So. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.



Example 14. Compute $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ for $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Example: $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$= \dfrac{\sin t}{t}$ where $t = x^2+y^2 \geq 0$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{t\to 0+} \frac{\sin t}{t} = 1$ L'Hospital's Rule