§1.7 Multivariable functions

We have a quick introduction on concepts/terminologies about functions and continuity. More details in MATH 3150 ⟨Real Analysis⟩ class, more generally in MATH 5121 ⟨Topology⟩ class. (If you have difficulty on reading section §1.7 in the book, do not worry too much. You can be good at in Calculus 3.)

A **multivariable function** $\vec{f} = \langle f_1, f_2, \ldots, f_m\rangle$ from $\mathbb{R}^3$ to $\mathbb{R}^m$ is defined by component functions $f_i(x, y, z)$ for $i = 1, 2, ..., m$.

The **domain** of $\vec{f}$ is a subset D of $\mathbb{R}^3$. (Similarly for $\mathbb{R}^n$).

Each f_i is a real value multivariable function from $\mathbb{R}^3$ to $\mathbb{R}$.

For example, a multivariable function f from $\mathbb{R}^3$ to $\mathbb{R}$ can be defined as $f(x, y, z) = e^x+y^3+\ln z$. In this function, $f(1, 2, 1) = e + 8$ is a real number. The domain is $\{(x, y, z|z > 0)\}$

level set where $f(x, y, z) = b$ is the set of points (x, y, z) such that $f(x, y, z) = b$.

Example 1. The level set of $\vec{f}(x, y) = (x^2, y^3 + 1, x + y + 1)$ where $\vec{f} = \vec{0}$ is $(0, -1)$.

A function $L : \mathbb{R}^3 \to \mathbb{R}$ is called **linear** if and only if $L(x, y, z) = ax + by + cz$ for some real numbers a, b, c.

A function $L : \mathbb{R}^3 \to \mathbb{R}$ is called **affine linear** if and only if $L(x, y, z) = ax + by + cz + d$ for some real numbers a, b, c, d.

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if each component function is linear.

Example 2. $F(x, y) = (3x - y, x + y, -6x + 2y - 1)$ is an affine linear function.

An **elementary function** is finite addition, subtraction, multiplication, division, or composition of x^a, b^x, $\ln(z)$, $\sin y$, ...

For example, $f(x, y, z) = \dfrac{e^x + y^3 + \ln z}{\tan(x + y^2)}$ is an elementary function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

All elementary functions are continuous.

Not all functions are continuous.

Theorem. Continuity

If $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then
1. For every open subset $U \subset \mathbb{R}^m$, the set $\vec{f}^{-1}(U)$ is open in $\mathbb{R}^n$.
2. For every closed subset $C \subset \mathbb{R}^m$, the set $\vec{f}^{-1}(C)$ is closed in $\mathbb{R}^n$.

In fact, if one of the statement is true, then $\vec{f}$ is continuous.

The intersection of a finite number of open subsets is open, and the intersection of any number of closed subsets is closed.

Example 3. Open balls and closed balls in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

In $\mathbb{R}^3$, let $P = (a, b, c)$ be the center of the ball.

Open balls: $B(P, r) = \{(x, y, z)|(x - a)^2 + (y - b)^2 + (z - c)^2 < r^2\}$.

Closed balls: $\overline{B(P, r)} = \{(x, y, z)|(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2\}$.

Boundary: $\overline{B(P, r)} = \{(x, y, z)|(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2\}$.

Example 4. $f(x, y, z) = x^2y + y\sqrt{z}$. The domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3|z \geq 0\},$$

which is closed in $\mathbb{R}^3$. All points in domain D satisfying $1 \leq f(x, y, z) \leq 3$ is denoted by $f^{-1}([1, 3])$, called the preimage of $[1, 3]$. Since $[1, 3]$ is closed, the preimage $f^{-1}([1, 3])$ is closed in D.

Theorem. Extreme Value Theorem

A continuous function from a compact subset of $\mathbb{R}^n$ into $\mathbb{R}$ has both a minimum and maximum value.

In $\mathbb{R}^n$, compact means closed and bounded. (There is a more general definition of compact subset in Topology.)

In particular, when $n = 1$, a continuous function on a closed, bounded interval $[a, b]$ has both a minimum value and a maximum value.

Example 5. Let E be the set of those $(x, y, z) \in R^3$ such that $x + 2y + z = 3$. Then, E is not compact, because it is not bounded.

Example 6. $f(x, y, z) = x^2+y^3-2z$, and E is the set of those (x, y, z) such that $x^2+y^2+z^2 = 4$. Then E is a sphere of radius 2, which is closed and bounded, so E is compact. Since f is continuous, it achieves its maxima and minima on E.

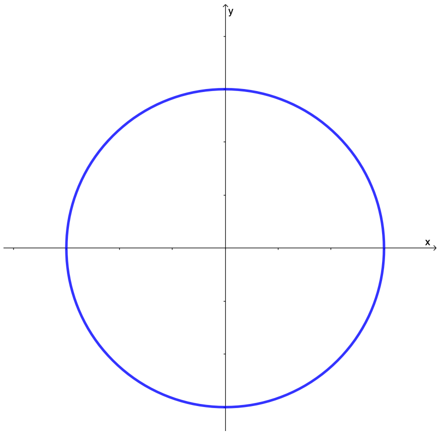
Theorem. Intermediate Value Theorem

If $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then for every connect subset $E \subset \mathbb{R}^n$, the set $f(E)$ is connect in $\mathbb{R}^n$.

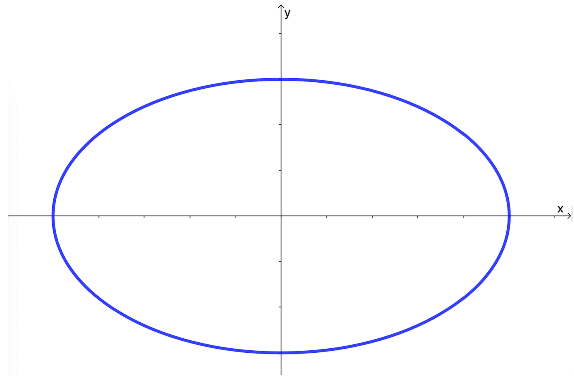
§1.8 Graphing surfaces

Review: Curves in $\mathbb{R}^2$:

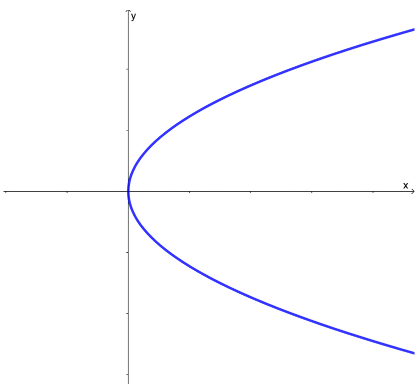
1. **circle**, $x^2 + y^2 = r^2$



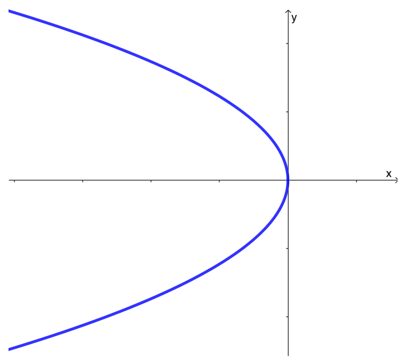
2. **ellipse**, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



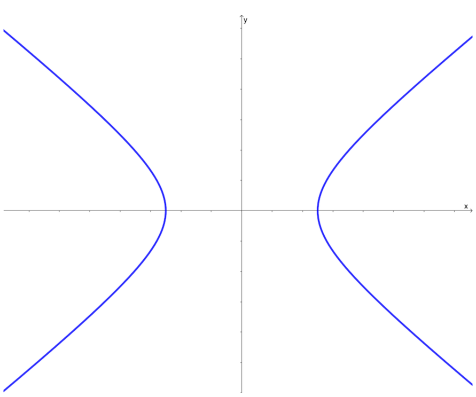
3. **parabola**, $y^2 = 4ax$



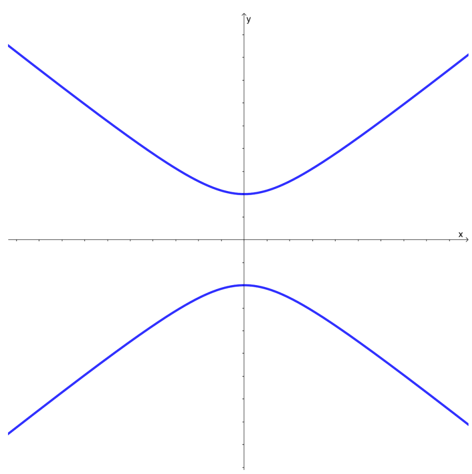
$y^2 = -4ax$ for $a > 0$



4. **hyperbola**, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



$-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Definition.

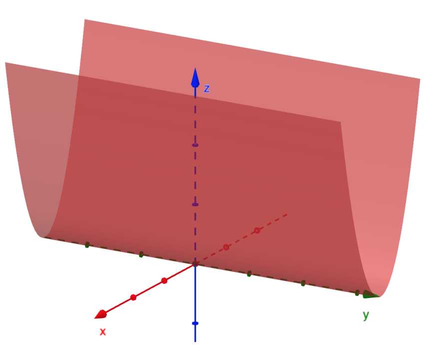
A **cylinder** in $\mathbb{R}^3$ is a surface that consists of all lines (called **rulings**) parallel to a given line and pass through a given plane curve. (The equation only includes two variables.)

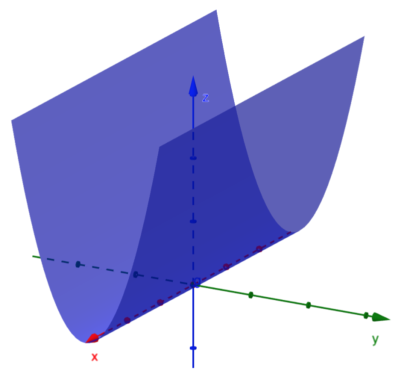
To Sketch the **graph of a surface** by hand, we determine x,y,z **cross sections** of the surface with planes parallel to the coordinate planes. (e.g., z-cross sections are intersections with $z = a$, planes parallel to xy-plane.)

Use https://www.geogebra.org/3d to look at those graphs.

Example 1. Sketch the graph of the surface $z = x^2$ and $z = y^2$.

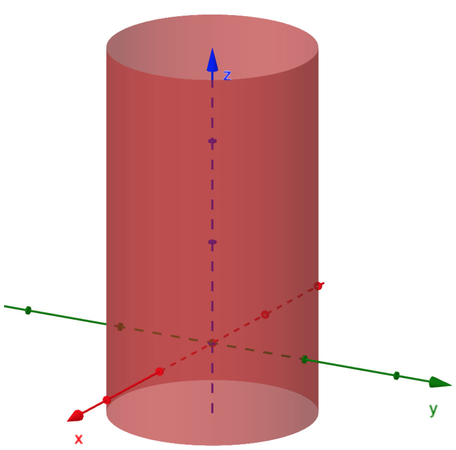
To study $z = x^2$, look at y-cross sections with $y = a$, planes parallel to xz-plane.

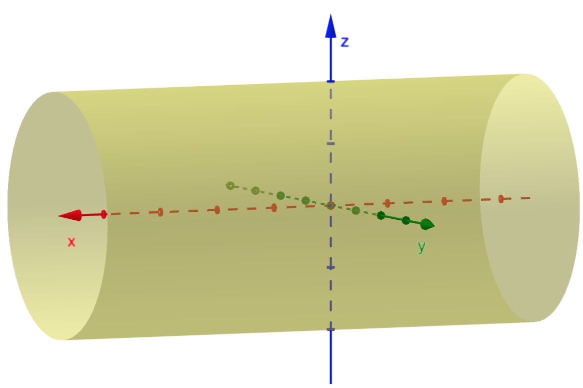




Example 2. Sketch the graph of the surfaces $x^2 + y^2 = 4$ and $y^2 + z^2 = 4$.

To study $x^2 + y^2 = 4$, look at z-cross sections with $z = a$, planes parallel to xy-plane.





Definition.

A **quadric surface** is the graph of a second-degree equation in three variables x, y and z. A general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

With translation and rotation, there are **two** standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + IZ = 0$$

Example 3. (**Ellipsoid**) Sketch the quadric surface with equation

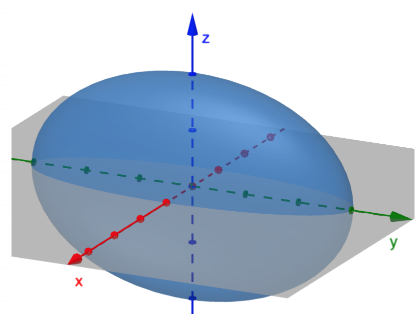
$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

intersect xy-plane: Substitute $z=0$ $\Rightarrow x^2 + \frac{y^2}{9} = 1$ ellipse

substitute $z=k$ $\Rightarrow x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$ ellipse if $1 - \frac{k^2}{4} > 0$ (or $-2<k<2$)

Intersect yz-plane: substitute $x=k \Rightarrow \frac{y^2}{9} + \frac{z^2}{4} = 1-k^2$. ellipse if $-1<k<1$

Intersect xz-plane: substitute $y=k \Rightarrow x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9}$. ellipse if $-3<k<3$



Example 4. (**Elliptic paraboloid**) Sketch the quadric surface with equation

$$z = x^2 + 4y^2.$$

$z = x^2 + 4y^2.$

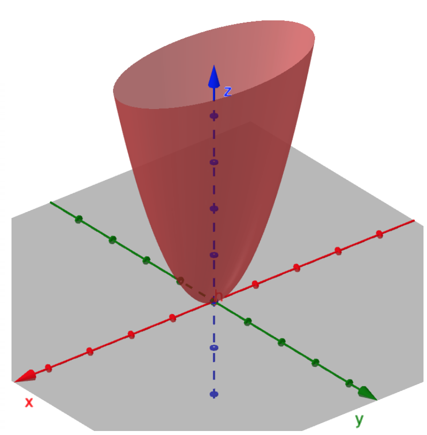
Horizontal

xy-plane parallel. $z=k \Rightarrow x^2+4y^2=k \Rightarrow$ ellipse if $k>0$

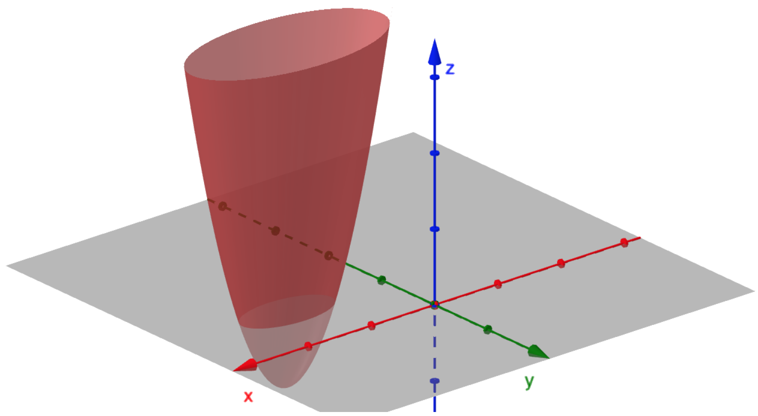
Vertical

yz-plane parallel: $x=k \Rightarrow z = k^2+4y^2 \Rightarrow$ parabola

xz-plane: $y=k \Rightarrow z = x^2+4k^2 \Rightarrow$ parabolla



Example 5. Sketch the quadric surface with equation $z + 2 = (x-3)^2 + 4(y+2)^2$.



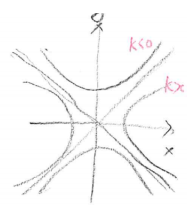
Example 6. (**Hyperbolic paraboloid**) Sketch the quadric surface with equation

$$z = x^2 - 4y^2.$$

$z = x^2 - 4y^2.$

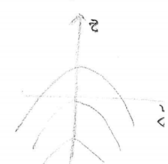
Horizontal

xy-plane $z=k$ $\Rightarrow$ $x^2-4y^2=k$ hyperbolas if $k\neq 0$; Two lines if $k=0$

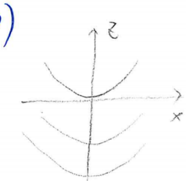


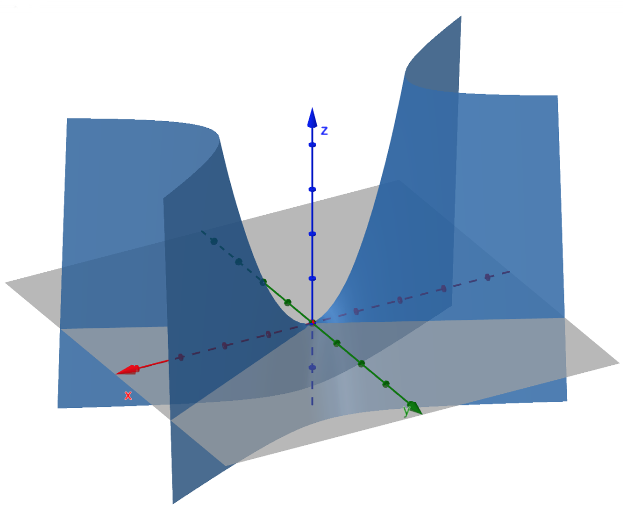
Vertical:

yz-plane $x=k$ $\Rightarrow$ $z=k^2-4y^2$ parabolas (open down)



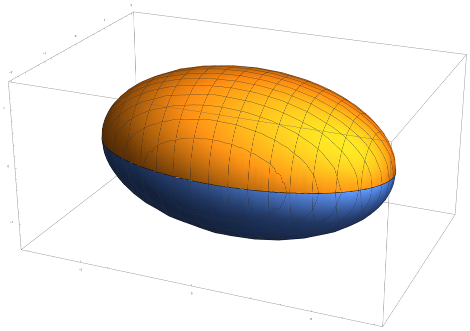
xz-plane $y=k$ $\Rightarrow$ $z=x^2-4k^2$ parabolas (open up)





1. Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

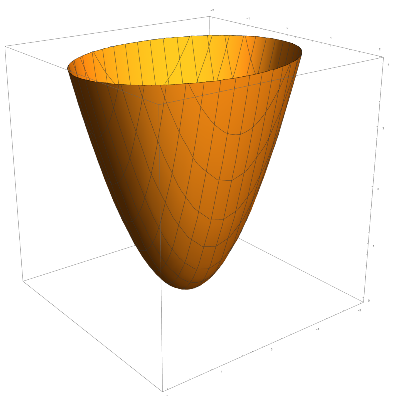
All traces are ellipses.



2. Elliptic Paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Horizontal traces are ellipses.

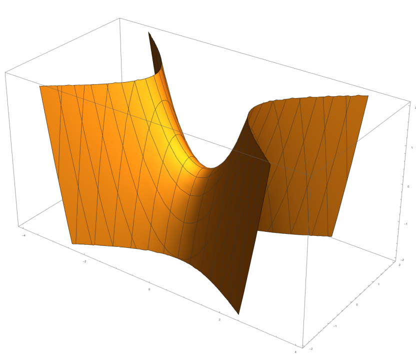
Vertical traces are parabolas.



2'. Hyperbolic Paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

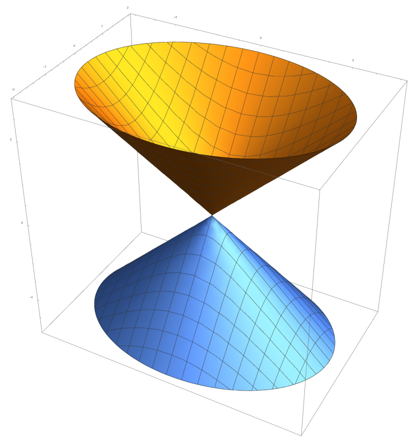
Horizontal traces are hyperbolas.

Vertical traces are parabolas.



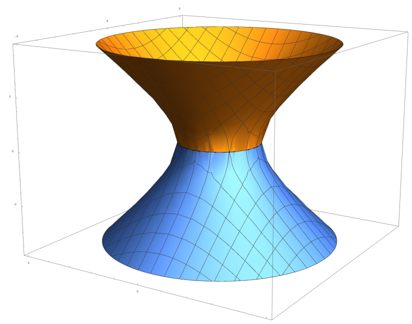
3. Cone: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Horizontal traces are ellipses.
Vertical traces hyperbolas, or pairs or lines.



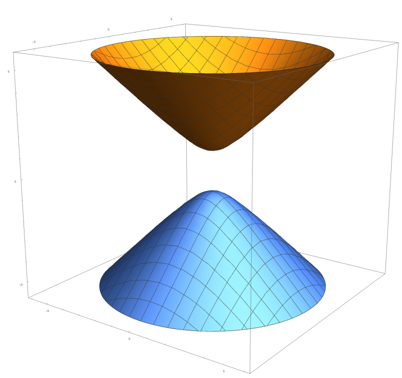
3'. Hyperboloid1: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

Horizontal traces are ellipses.
Vertical traces are hyperbolas.



3". Hyperboloid2: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+1$

Horizontal traces in are ellipses, or a point.
Vertical traces are hyperbolas



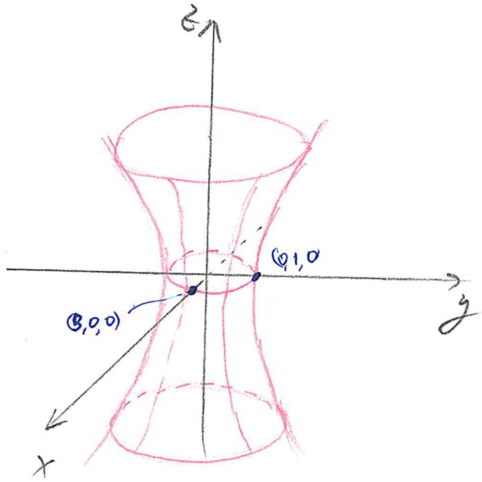
Example 7. (**Hyperboloid of one sheet**) Sketch the quadric surface with equation

$$\frac{x^2}{9} + y^2 - \frac{z^2}{4} = 1.$$

z-**cross sections:** Horizontal intersections with $z = a$ give $\frac{x^2}{9} + y^2 = 1 + \frac{a^2}{4}$, which are ellipses.

x-**cross sections:** Vertical intersections with (parallel yz-planes) $x = c$ give $y^2 - \frac{z^2}{4} = 1 - \frac{a^2}{9}$, which are parabolas.

y-**cross sections:** Vertical intersections with (parallel xz-planes) $y = c$ give $\frac{x^2}{9} - \frac{z^2}{4} = 1 - a^2$, which are parabolas.



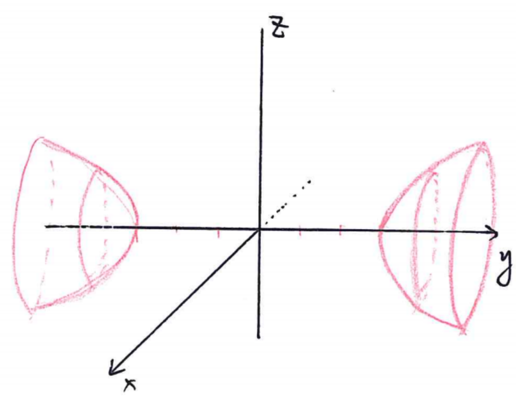
Example 8. Identify and sketch the surface $6x^2 - y^2 + 3z^2 + 9 = 0$

$y^2 = 6x^2 + 3z^2 + 9$

$\frac{y^2}{9} = \frac{x^2}{3/2} + \frac{z^2}{3} + 1$

- Hyperboloid with 2 sheets
- Axis - y. symmetry

$\frac{y^2}{9} \geq 1$ $|y| \geq 3$



Example 9. Classify the quadratic surface $x^2 + 3z^2 - 4x - y + 7 = 0$

$(x^2-4x+4)+3z^2-y+3=0$ $\qquad$ $(x-2)^2+3z^2=y-3$

parallel move $x^2+3z^2=y$ to $(2,3,0)$