§1.6 Functions of a single variable

1. Limits and continuous.

A **vector function** (or single-variable function into $\mathbb{R}^n$) is a function whose **domain** is subset of $\mathbb{R}$ and whose **range** is subset of vectors in $\mathbb{R}^n$.

We study examples when $n = 2$ or $n = 3$. Higher dimension functions are similarly.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

The functions $f(t), g(t), h(t)$ are called **component functions**.

Example 1. $\vec{r}(t) = \langle \sqrt{t-1}, t^2 - t, \ln(4-t) \rangle$

The component functions are $f(t) = \sqrt{t-1}$, $g(t) = t^2 - t$, and $h(t) = \ln(4-t)$.
For the domain, $t - 1 \geq 0$ and $4 - t > 0$. So, the domain is $1 \leq t < 4$.

Definition.

The **limit** of a vector function is defined by taking the limits of its component functions,

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle.$$

A vector function $\vec{r}(t)$ is **continuous** at $t = a$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Example 2. Find the limit $\lim_{t\to 0} \vec{r}(t)$ for vector function $\vec{r}(t) = \langle \ln(1-t), t^2 + 2, \frac{\sin t}{t} \rangle$.

$\lim_{t\to 0} \vec{r}(t) = \langle \lim_{t\to 0} \ln(1-t), \lim_{t\to 0}(t^2+2), \lim_{t\to 0} \frac{\sin t}{t} \rangle = \langle 0, 2, 1 \rangle$.
We used L'Hospital's rule for the last limit.

Definition.

Suppose $\vec{r}(t)$ is *continuous* on an interval I. Then set of points $(f(t), g(t), h(t))$ in $\mathbb{R}^3$ is called a **space curve**. The equations

$$x = f(t), y = g(t), z = h(t)$$

are **parametric equations** of the curve and t is a **parameter**.

Example 3. Describe the curve defined by the vector function $\vec{r}(t) = (1 + 6t)\vec{i} + (2 - 3t)\vec{j} + (3 - t)\vec{k}$.

A line passing through $(1, 2, 3)$ parallel to the vector $\langle 6, -3, -1\rangle$.

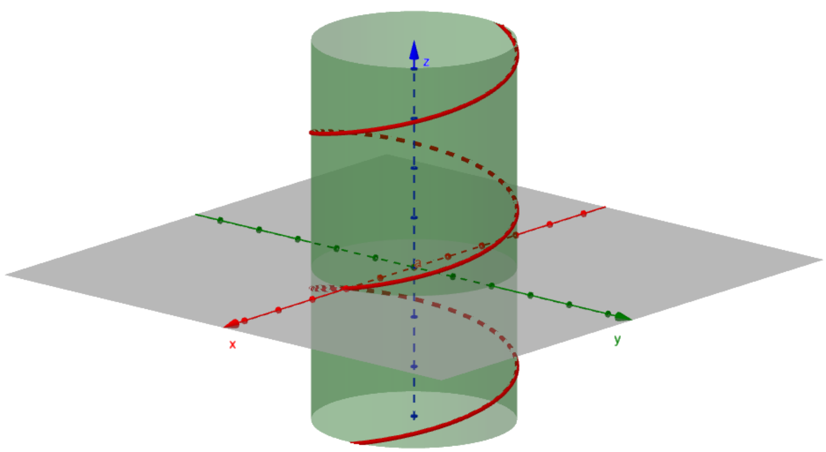
Example 4. Find a vector equation (standard parameterization) and parametric equations for the line segment that joins the points P$(3, -2, 4)$ and Q$(-1, 2, 4)$.

From §1.4, we know the line segment is

$$(x, y, z) = P + t(\overrightarrow{PQ}) = (3, -2, 4) + t(-4, 4, 0) \text{ for } 0 \leq t \leq 1.$$

So, $x(t) = 3 - 4t$, $y(t) = -2 + 4t$, $z(t) = 4$ for $0 \leq t \leq 1$.

Example 5 (Helix). Sketch the curve whose vector equation is $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$.



Projection onto xy-plane is $\langle \cos t, \sin t, 0\rangle$, which is a circle.

Example 6. Find a vector function (**parametrization**) for the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 3$ in $\mathbb{R}^3$.

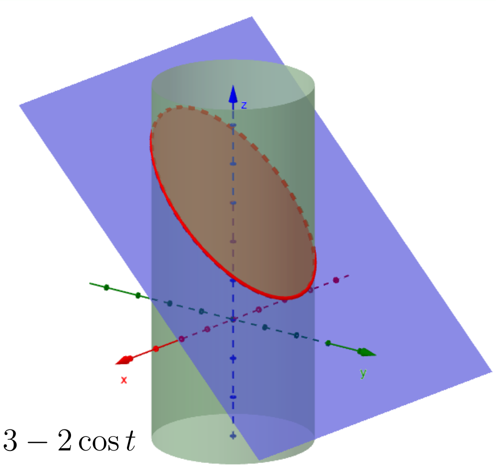
$$\frac{x^2}{2^2} + \frac{y^2}{2^2} = 1$$

Let $\frac{x}{2} = \sin t$ and $\frac{y}{2} = \cos t$ for $0 \leq t \leq 2\pi$.

So, $x = 2\sin t$ and $y = 2\cos t$.

Substitute to $y + z = 3$, we have $2\cos t + z = 3$, which implies $z = 3 - 2\cos t$.

The intersection has parametric equation



$$x = 2\sin t; y = 2\cos t; z = 3 - 2\cos t$$

Example 7. Find the projection of the space curve $\vec{r}(x) = \langle t, 2t, 3t + 2t^2 \rangle$ onto the coordinate planes.

The projection onto xy-plane is $\langle t, 2t, 0 \rangle$.
The projection onto xz-plane is $\langle t, 0, 3t + 2t^2 \rangle$.
The projection onto yz-plane is $\langle 0, 2t, 3t + 2t^2 \rangle$.

Example 8. Find the intersection points of the space curve $\vec{r}(x) = \langle t, 2t, 3t + 2t^2 \rangle$ and the paraboloid $z = x^2 + y^2$.

Substitute $x = t, y = 2t, z = 3t + 2t^2$ to the paraboloid $z = x^2 + y^2$, we have $3t + 2t^2 = t^2 + (2t)^2$. Solve the equation, we have $t = 0$ or $t = 1$.
For $t = 0$, we have an intersection point $(0, 0, 0)$.
For $t = 1$, we have an intersection point $(1, 2, 5)$.

2. Derivatives and integrals

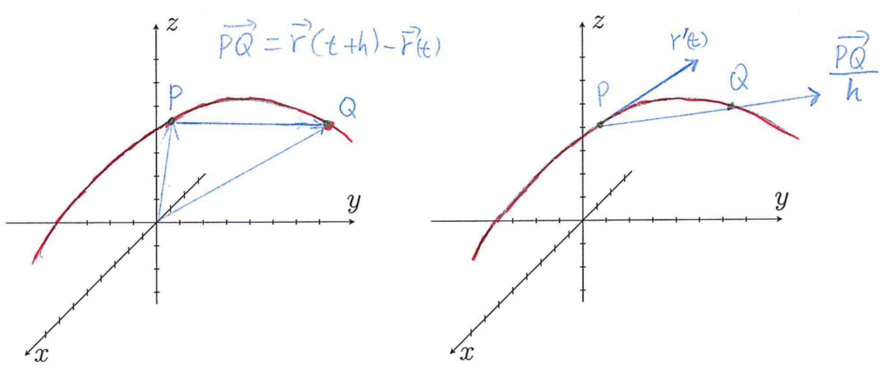
Definition.

The **derivative** of a vector function $\vec{r}(t)$ is defined as

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

The vector $\vec{r}\,'(t)$ is also called the **tangent vector** to the curve.
The **unit tangent vector** is defined as

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$



Theorem. Computation formula

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ for differentialble functions $f(t), g(t), h(t)$, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

Example 9. (1) Find the derivative of $\vec{r}(t) = (t^2 - 2t^3)\vec{i} + te^{1-t}\vec{j} - \sin(3t)\vec{k}$.

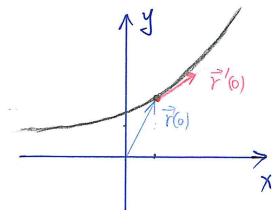
(2) Find the unit tangent vector at the point where $t = 0$.

(1). $\vec{r}'(t) = (2t - 6t^2)\vec{i} + (e^{1-t} - t(e^{1-t})\vec{j} - 3\cos(3t))\vec{k}$.
(2). $\vec{r}'(0) = \langle 0, e, -3\rangle$. So the unit vector is

$$\vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{1}{\sqrt{e^2+9}}\langle 0, e, -3\rangle.$$

Example 10. For the curve $\vec{r}(t) = \ln(2t + e)\vec{i} + (t + 2)\vec{j}$, find and sketch the position vector $\vec{r}(0)$ and the tangent vector $\vec{r}\,'(0)$.

$\vec{r}'(t) = \langle \frac{2}{2t+3},\ 1\rangle$.
So, $\vec{r}(0) = \langle 1, 2\rangle$ and
$\vec{r}'(0) = \langle \frac{2}{e}, 1\rangle$



Example 11. Find parametric equations for the tangent line to the curve with parametric equations

$$x = t + 3\cos 2t, \quad y = 2te^{t^2}, \quad z = t^3 + 1$$

at point $(3, 0, 1)$.

$$\vec{r}\,'(t) = \langle 1 - 6\sin 2t,\ 2e^{t^2} + 2t(2t)e^{t^2},\ 3t^2\rangle$$

The position vector is $\vec{r}_0 = \vec{r}(0) = \langle 3, 0, 1\rangle$.
From $z = 1$, we have $t^3 + 1 = 1$, so $t = 0$.
The direction vector is $\vec{r}\,'(0) = \langle 1, 2, 0\rangle$.
The tangent line has equation $\vec{r}(t) = \langle 3, 0, 1\rangle + t\langle 1, 2, 0\rangle$.
The parametric equation is $x = 3 + t$, $y = 2t$, $z = 1$.

Definition.

The **second derivative** of $\vec{r}(t)$ is the derivative of the first derivative. That is $\vec{r}\,'' = (\vec{r}\,')'$

Example 12. Find the second derivative of $\vec{r}(t) = \langle \sin(2t), \ln(t), t^3\rangle$.

$\vec{r}\,'(t) = \langle 2\cos(2t), \frac{1}{t}, 3t^2\rangle$
$\vec{r}\,'' = \langle -4\sin 2t, -t^{-2}, 3t\rangle$

Theorem. Differentiation Rules:

Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}\,'(t)+\vec{v}\,'(t)$ ······ linear property
2. $\dfrac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$ ······ linear property
3. $\dfrac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}\,'(t)$ ········· product rule
4. $\dfrac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}\,'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}\,'(t)$ ········· product rule
5. $\dfrac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}\,'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}\,'(t)$ ········· product rule
6. $\dfrac{d}{dt}[\vec{u}(f(t))] = \vec{u}\,'(f(t))f'(t)$ ············ chain rule

Example 13. Find the derivative $\dfrac{d}{dt}(\vec{a}(t)\cdot\vec{b}(t))$ for $\vec{a}=\langle t^3+1, t^2, \pi\rangle$ and $\vec{b}(t)=\langle 4, e^t, t+2\rangle$.

$$\begin{aligned}\frac{d}{dt}(\vec{a}(t)\cdot\vec{b}(t)) &= \vec{a}'\cdot\vec{b}(t)+\vec{a}\cdot\vec{b}'(t)\\ &= \langle 3t^2, 2t, 0\rangle\cdot\langle 4, e^t, t+2\rangle+\langle t^3+1, t^2, \pi\rangle\cdot\langle 0, e^t, 1\rangle\\ &= 12t^2+2te^t+t^2e^t+\pi\end{aligned}$$

Definition.

The **definite integral** of a continuous vector function $\vec{r}(t)$ can be defined as

$$\int_a^b \vec{r}(t)dt = \lim_{n\to\infty}\sum_{i=1}^n \vec{r}(t_i)\Delta t.$$

We also have an easy computation formula (Theorem) for the integral of $\vec{r}(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k}$,

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i}+\left(\int_a^b g(t)dt\right)\vec{j}+\left(\int_a^b h(t)dt\right)\vec{k}$$

The **Fundamental Theorem of Calculus** also can be extended to vector calculus:

$$\int_a^b \vec{r}(t)dt = \vec{R}(b)-\vec{R}(a)$$

where $\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$, i.e., $\vec{R}'(t) = \vec{r}(t)$

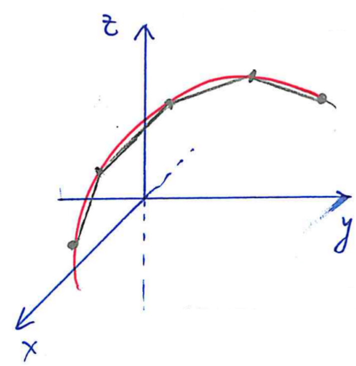
Example 14. If $\vec{r}(t) = (2\sin t)\vec{i} + (e^t + 1)\vec{j} + (8t^3)\vec{k}$, calculate $\int \vec{r}(t)dt$ and $\int_0^{\pi/2} \vec{r}(t)dt$.

$$\int \vec{r}(t)\,dt = \left(\int 2\sin t\,dt\right)\vec{i} + \left(\int e^t+1\,dt\right)\vec{j} + \left(\int 8t^3\,dt\right)\vec{k}$$
$$= (-2\cos t + C_1)\vec{i} + (e^t + t + C_2)\vec{j} + (2t^4 + C_3)\vec{k}$$
$$= (-2\cos t)\vec{i} + (e^t + t)\vec{j} + (2t^4)\vec{k} + \vec{C}$$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \vec{R}\left(\tfrac{\pi}{2}\right) - \vec{R}(0)$$
$$= \left(e^{\pi/2} + \tfrac{\pi}{2}\right)\vec{j} + 2\left(\tfrac{\pi}{2}\right)^4\vec{k} - (-2\vec{i} + \vec{j})$$
$$= 4\vec{i} + \left(e^{\pi/2} + \tfrac{\pi}{2} - 1\right)\vec{j} + \frac{\pi^4}{8}\vec{k}$$

3 Arc length and curvature

Suppose a curve has the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \le t \le b$, or the parametric equation $x = f(t),\ y = g(t),\ z = h(t)$, where $f'(t), g'(t), h'(t)$ are continuous.



Definition.

If the curve is traversed exactly once as t increases from a to b, then its **length** is

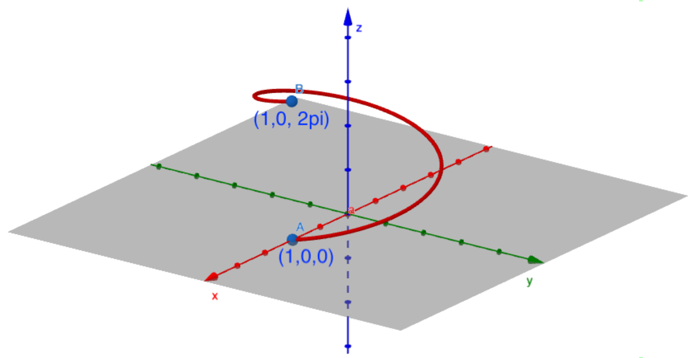
$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

A compact formula for the arc length is

$$L = \int_a^b |\vec{r}\,'(t)|dt$$

Example 15. Find the length of the arc of the circular helix with vector equation $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$
$|\vec{r}'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$
The length of the arc is
$L = \int_0^{2\pi} |\vec{r}'(t)|dt = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi$



Definition.

The **arc length function** $s(t)$ is the length of the curve between $\vec{r}(a)$ and $\vec{r}(t)$ defined by

$$s(t) = \int_a^t |\vec{r}\,'(u)|du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

From the Fundamental Theorem of Calculus, differentiate both sides, we have

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

It is often useful to parametrize a curve with respect to arc length s because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

Example 16. Reparametrize the helix $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

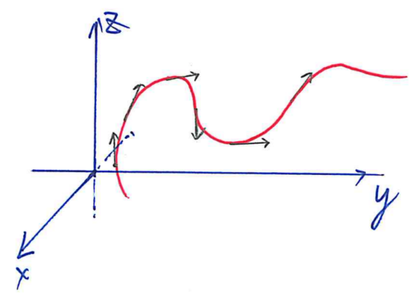
$$\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{2}$$

So, $s = s(t) = \int_0^t |\vec{r}'(u)|du = \sqrt{2}t$.
$t = s/\sqrt{2}$.
So, $\vec{r}(t) = (\cos(s/\sqrt{2}))\vec{i} + (\sin(s/\sqrt{2}))\vec{j} + (s/\sqrt{2})\vec{k}$

Definition.

* Let T be the unit tangent vector and s be the arc length.
The **curvature** of a curve is

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$



1. Curvature measure how quickly the curve change directions.
2. The curvature of a curve can be computed by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$$

Example 17. *Show that the curvature of a circle of radius a is $1/a$.

$\vec{r}'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$ $\qquad$ $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{r}'(t)}{a} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$T'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$

$K(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{a}$

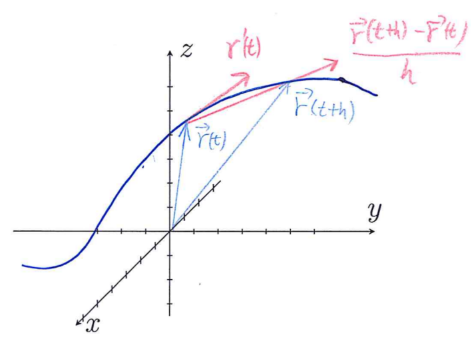
Theorem.

*The curvature of the curve given by the vector function $\vec{r}$ is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

4. Motion in space: Velocity and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\vec{r}(t)$.



Definition.

The **velocity vector** $\vec{v}(t)$ at time t is

$$\vec{v}(t) = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

The velocity vector is also the tangent vector.
The **speed** of the particle at time is the magnitude of the velocity vector, that is, $|\vec{v}(t)|$.
The **acceleration** of the particle is defined as the derivative of the velocity,

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Example 18. The position vector of an object moving in a plane is given by

$$\vec{r}(t) = \langle t^3, (1/2)t \rangle.$$

Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

Velocity $\vec{V}(t) = \vec{r}\,'(t) = \langle 3t^2, \frac{1}{2} \rangle$

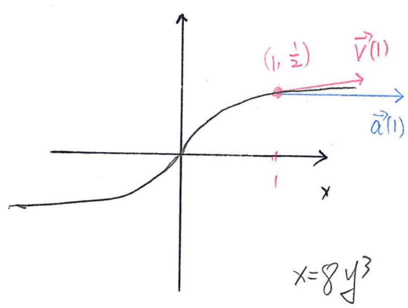
Speed $|\vec{V}(t)| = \sqrt{(3t^2)^2 + (\frac{1}{2})^2} = \sqrt{9t^4 + \frac{1}{4}}$

Acceleration $\vec{a}(t) = \vec{V}\,'(t) = \langle 6t, 0 \rangle$

When $t=1$, $\vec{V}(1) = \langle 3, \frac{1}{2} \rangle$

$|\vec{V}(1)| = \sqrt{9 + \frac{1}{4}} = \frac{\sqrt{37}}{2}$

$\vec{a}(1) = \langle 6, 0 \rangle$



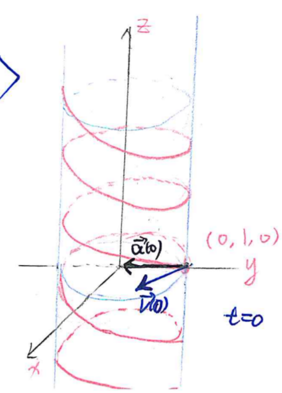
Example 19. Find the velocity, acceleration, and speed of a particle with position vector

$$\vec{r}(t) = \langle \sin t, \cos t, t \rangle$$

Velocity: $\vec{V}(t) = \vec{r}\,'(t) = \langle \cos t, -\sin t, 1 \rangle$

Acceleration: $\vec{a}(t) = \vec{V}\,'(t) = \langle -\sin t, -\cos t, 0 \rangle$

Speed: $|\vec{V}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$



Example 20. A moving particle starts at an initial position $\vec{r}(0) = \langle 0, 0, 1 \rangle$ with initial velocity $\vec{v}(0) = \vec{i} + \vec{k}$. Its acceleration is $\vec{a}(t) = \langle 2t, 1, 6t \rangle$. Find its velocity and position at time t.

Step 1 (Velocity)

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle t^2, t, 3t^2 \rangle + \vec{C_1}$

$\vec{v}(0) = \langle 1, 0, -1 \rangle \Rightarrow \langle 1, 0, -1 \rangle = \langle 0, 0, 0 \rangle + \vec{C_1}$

$\Rightarrow \vec{C_1} = \langle 1, 0, -1 \rangle$

So $\vec{v}(t) = \langle t^2+1, t, 3t^2-1 \rangle$

Step 2 (position)

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{t^3}{3} + t, \frac{t^2}{2}, t^3 - t \rangle + \vec{C_2}$

$\vec{r}(0) = \langle 0, 0, 1 \rangle \Rightarrow \langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle + \vec{C_2}$

$\Rightarrow \vec{C_2} = \langle 0, 0, 1 \rangle$

$\Rightarrow \vec{r}(t) = \langle \frac{t^3}{3} + t, \frac{t^2}{2}, t^3 - t + 1 \rangle$

Example 21. A particle has position function $\vec{r}(t) = \langle t^2 + 1, 3t, t^2 - 4t \rangle$. When is the speed a minimum?

Step 1: Find speed function.

$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 3, 2t-4 \rangle$

Speed: $|\vec{v}(t)| = \sqrt{(2t)^2 + 3^2 + (2t-4)^2} = \sqrt{8t^2 - 16t + 25}$

Step 2 Find critical points

$\frac{d}{dt}|\vec{v}(t)| = \frac{1}{2}(8t^2 - 16t + 25)^{-\frac{1}{2}} \cdot (16t - 16) \cdot$ chain rule

critical points when $16t - 16 = 0 \Rightarrow t = 1$

Step 3: 1st derivative test: $\frac{d}{dt}|\vec{v}(0)| < 0$ | $\frac{d}{dt}|\vec{v}(2)| > 0$

decrease | increase

$\Rightarrow$ when $t=1$, the speed is a minimum.