§1.5 Cross product

Definition.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

Theorem.

The cross product $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$ following the right-hand rule.

(Proof: Check $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ and $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$.)

The cross product $\vec{a} \times \vec{a} = \vec{0}$ for any $\vec{a} \in \mathbb{R}^3$.

Remark: The cross product is **only** defined in $\mathbb{R}^3$ and $\mathbb{R}^7$. All definitions in §1.1-1.4 are for $\mathbb{R}^n$.

• To make the definition easier to remember, we use the notation of **determinants** from linear algebra.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example 1. Compute the determinants $\begin{vmatrix} 4 & 6 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{vmatrix}$

$$\begin{vmatrix} 4 & 6 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{vmatrix} = 4 \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} - 6 \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} 5 & 2 \\ 0 & 2 \end{vmatrix} = 4(-6) - 6(-5) + 2(10) = 26$$

Using the notation of determinant, the cross product of $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Example 2. If $\vec{a} = \langle 1, 2, 3 \rangle$ and $\vec{b} = \langle 2, 4, -3 \rangle$, find the cross product $\vec{a} \times \vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & -3 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}\vec{k} = -18\vec{i} + 9\vec{j}$$

Theorem.

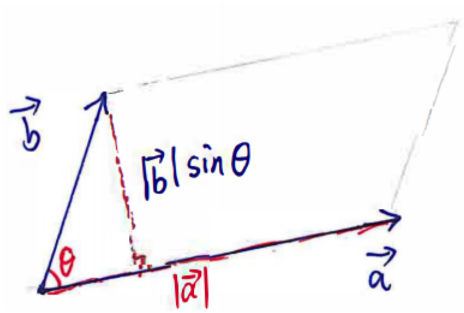
If θ is the angle between $\vec{a}$ and $\vec{b}$, (so $0 \le \theta \le \pi$), then

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$

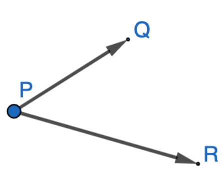
∗ Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a}\times\vec{b} = \vec{0}$.

∗ (**Geometric meaning**) The length(magnitude) of the cross product $\vec{a}\times\vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Area$=|\vec{a}|(|\vec{b}|\sin\theta)$



Example 3. Find a vector perpendicular to the plane that passes through the points $P(1,2,3)$, $Q(3,-2,2)$ and $R(-1,1,0)$.



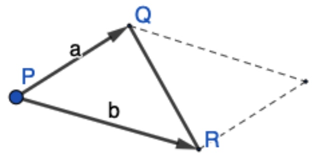
$$\overrightarrow{PQ} = \langle 3-1, 0-2, 2-3\rangle = \langle 2,-2,-1\rangle$$

$$\overrightarrow{PR} = \langle -1-1, 1-2, 0-3\rangle = \langle -2,-1,-3\rangle$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ -2 & -1 & -3 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -1 & -3 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -1 \\ -2 & -3 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix}\vec{k}$$

$$= 4\vec{i} + 8\vec{j} - 6\vec{k}$$

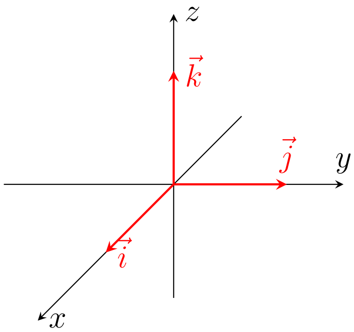
Example 4. Find the area of the triangle with vertices $P(1,2,3)$, $Q(3,0,2)$ and $R(-1,1,0)$.



$\overrightarrow{PQ} = \langle 2, -2, -1\rangle$ and $\overrightarrow{PR} = \langle 2, -1, -3\rangle$.

Triangle Area=$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{4^2 + 8^2 + 6^2} = \sqrt{28}$

Cross products of standard basis vectors.



$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

$$\vec{j} \times \vec{i} = -\vec{k} \quad \vec{k} \times \vec{j} = -\vec{i} \quad \vec{i} \times \vec{k} = -\vec{j}$$

The cross product is neither commutative nor associative, i.e.,

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}, \qquad \vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Theorem.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors and k is a scalar, then

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = a \times (k\vec{b})$

(3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

(5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

(6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

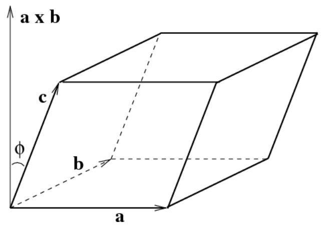
Scalar Triple Products

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Theorem.

The **volume** of the **parallelepiped** determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$



$*$ In particular, if $V = 0$, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ are on the same plane, i.e., they are **coplanar**.

Example 5. Use the scalar triple product to show that the vectors $\vec{a} = \langle -3, 0, 6\rangle$, $\vec{b} = \langle 4, 1, -7\rangle$ and $\vec{c} = \langle 1, -2, -4\rangle$ are coplanar.

Compute $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} -3 & 0 & 6 \\ 4 & 1 & -7 \\ 1 & -2 & -4 \end{vmatrix} = 0$

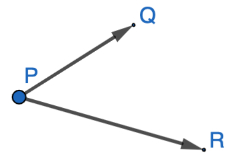
Example 6. Find the standard equation containing the line $(x, y, z) = (1, 2, 3) + t(1, 0, 2)$ and the point $R(2, 3, 2)$.

The direction vector of the line is $\vec{v} = \langle 1, 0, 2\rangle$. Another vector passing $(1, 2, 3)$ and R is $\vec{w} = (2, 3, 2) - (1, 2, 3) = \langle 1, 1, -1\rangle$.
The normal vector of the plane is

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \vec{k} = -2\vec{i} + 3\vec{j} + 1\vec{k}$$

The equation for the plane is $-2(x - 1) + 3(y - 2) + 1(z - 3) = 0$
The standard equation is $-2x + 3y + z - 7 = 0$

Example 7. Find an equation of the plane trough the points $P(1, 2, 3)$, $Q(2, 2, 5)$ and $R(3, 4, 1)$.



$\overrightarrow{PQ} = \langle 1, 0, 2\rangle$ and $\overrightarrow{PR} = \langle 2, 2, -2\rangle$.

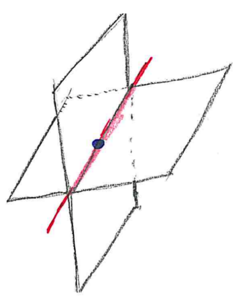
The normal vector of the plane is

$$\overrightarrow{PQ} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}\vec{k} = -4\vec{i} + 6\vec{j} + 2\vec{k}$$

The equation for the plane is $-4(x-1) + 6(y-2) + 2(z-3) = 0$

The standard equation is $-2x + 3y + z - 7 = 0$

Example 8. Find an equation of the intersection line by the two planes. $x + y + z = 3$ and $2x + 3y + z = 8$.



Step 1. Find a point on the line

Choose a value for z. for example Let $z=0$

Then $\begin{cases} x+y=3 \\ 2x+3y=8 \end{cases} \Rightarrow \begin{cases} x+y=3 \\ y=2 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}$

The point is $(1, 2, 0)$

Step 2. Find a direction vector.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -2\vec{i} + \vec{j} + \vec{k}$$

So an equation for the line is $\dfrac{x-1}{-2} = \dfrac{y-2}{1} = \dfrac{z}{1}$

Definition.

Consider a rigid rod, with one end at the point $A(x_0, y_0, z_0)$ and the other end at $B(x_1, y_1, z_1)$. If a force $\vec{F}$ is applied at B, then the **torque** produced by $\vec{F}$ around A is

$$\tau = \vec{d} \times \vec{F}$$

The torque vector comes out of the page by the right-hand rule. (Only for homework 37.)