§1.4 Lines, planes, and hyperplanes

▶ Review: Lines in $\mathbb{R}^2$.

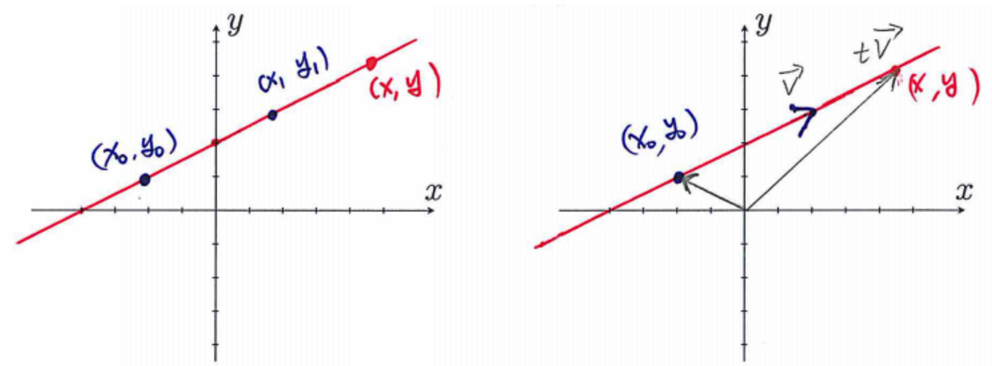
1. A line in $\mathbb{R}^2$ is determined by the slope k and a point (x_0, y_0) on the line.

$$y - y_0 = k(x - x_0)$$

2. A line in $\mathbb{R}^2$ is determined by two points (x_0, y_0) and (x_1, y_1) on the line.

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} \qquad \textbf{(Symmetric Equation)}$$

Example 1. The line in $\mathbb{R}^2$ defined by $y = 0.5x + 2$, passing two points $(-4, 0)$ and $(2, 3)$.



3. A line in $\mathbb{R}^2$ is determined by a position vector $\vec{r}_0 = \langle x_0, y_0 \rangle$ and a direction vector $\vec{v} = \langle a, b \rangle$. It can be written as a **vector equation**:

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

or written as

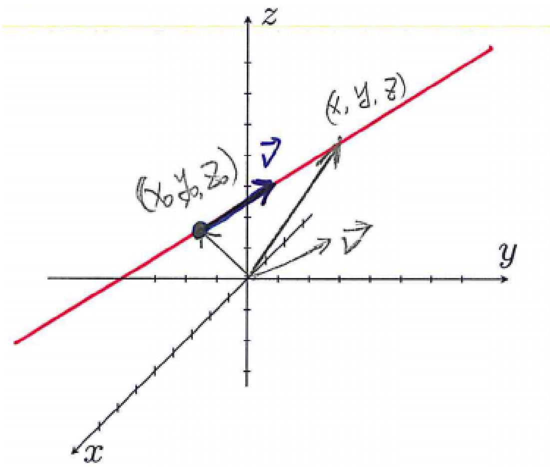
$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle a, b \rangle.$$

From two points, $\vec{v}$ is calculated by $\vec{v} = \langle x_1 - x_0, y_1 - y_0 \rangle$.

4. Equivalently, a line in $\mathbb{R}^2$ can be written as a **parametric equation**:

$$x = x_0 + at, \quad y = y_0 + bt.$$

▶ Lines in $\mathbb{R}^3$.



1. A line in $\mathbb{R}^3$ is determined by a position vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and a direction vector $\vec{v} = \langle a, b, c \rangle$ with **vector equation**

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.$$

t is called a **parameter**.
From two points (x_0, y_0, z_0) and (x_1, y_1, z_1), the direction vector $\vec{v}$ can be calculated by $\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.
2. Equivalently, a line in $\mathbb{R}^3$ can be written as **parametric equation**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. Equivalently, a line in $\mathbb{R}^3$ can be written as **symmetric equation**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Remark: All those three equations can be generalized to line in $\mathbb{R}^n$.

Example 2. Find a vector equation, parametric equation and symmetric equation for the line that passes through the point $(1, 2, 3)$ and is parallel to the vector $6\vec{i} - 3\vec{j} - \vec{k}$. Find two other points on the line.

Position vector $\vec{r}_0 = \langle 1, 2, 3 \rangle$ and direction vector $\vec{v} = \langle 6, -3, -1 \rangle$.

Vector equation is given by $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1, 2, 3 \rangle + t\langle 6, -3, -1 \rangle = \langle 1 + 6t, 2 - 3t, 3 - t \rangle$.

Parametric equation is $x = 1 + 6t; y = 2 - 3t; z = 3 - t$.

Symmetric equation is $\dfrac{x-1}{6} = \dfrac{y-2}{-3} = \dfrac{z-3}{-1}$.

Example 3. Find a vector equation for the line that passes through the point $(1, 2, 3)$ and is parallel to a line $x = 2 + 2t, y = 1 + 3t, z = 5 - 2t$.

Position vector $\vec{r}_0 = \langle 1, 2, 3\rangle$ and direction vector $\vec{v} = \langle 2, 3, -2\rangle$.

Vector equation is given by $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1, 2, 3\rangle + t\langle 2, 3, -2\rangle = \langle 1 + 2t, 2 + 3t, 3 - 2t\rangle$.

Example 4. Find a vector equation, parametric equation and symmetric equation for the line that passes through two points $(-5, 5, 4)$ and $(7, -1, 2)$

Position vector $\vec{r}_0 = \langle -5, 5, 4\rangle$ and direction vector $\vec{v} = \langle 12, -6, -2\rangle$.

Vector equation is given by $\vec{r} = \vec{r}_0 + t\vec{v} = \langle -5, 5, 4\rangle + t\langle 12, -6, -2\rangle = \langle -5 + 12t, 5 - 6t, 4 - 2t\rangle$.

Parametric equation is $x = -5 + 12t; y = 5 - 6t; z = 4 - 2t$.

Symmetric equation is $\dfrac{x+5}{12} = \dfrac{y-5}{-6} = \dfrac{z-4}{-2}$.

Theorem.

Two lines are **parallel** if and only if one direction vector is a scaler multiple of the the other direction vector.

Example 5. Are the line $x = 3+2t$, $y = 2-t$, $z = 8+3t$ and the line $\dfrac{x - x_0}{-4} = \dfrac{y - y_0}{2} = \dfrac{z - z_0}{-6}$ parallel?

The first line has direction vector $\vec{u} = \langle 2, -1, 3\rangle$.
The second line has direction vector $\vec{v} = \langle -4, 2, -6\rangle$.
Two lines are parallel since $\vec{v} = 2\vec{u}$.

Definition.

Standard parameterization of the line segment from point P (x_0, y_0, z_0) to point Q (x_1, y_1, z_1) is

$$(x, y, z) = (x_0, y_0, z_0) + t(\overrightarrow{PQ})$$

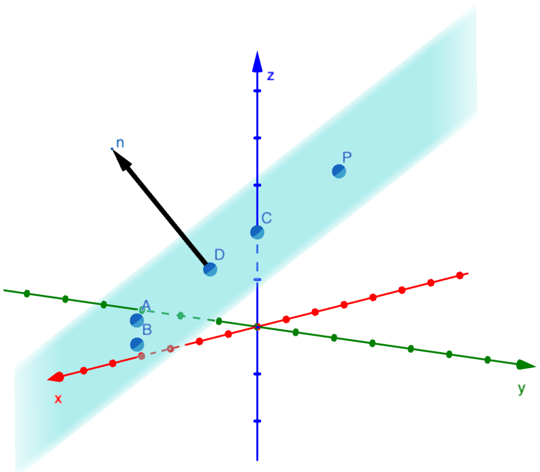
by **restricting** $0 \le t \le 1$. Here $\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$

Example 6. Give the parametric description of the **line segment** from the point $P(1, 2, 3)$ to the point $Q(4, 5, 7)$.

Direction vector is $\overrightarrow{PQ} = Q - P = \langle 3, 3, 4\rangle$.
The line segment is $(x, y, z) = (1, 2, 3) + t(3, 3, 4)$ for $0 \le t \le 1$.

▶ **Planes in $\mathbb{R}^3$.**

A plane in $\mathbb{R}^3$ is determined by a point P (with position vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$) on the plane and a vector $\vec{n} = \langle a, b, c \rangle$ that is orthogonal (perpendicular) to the plane.



For any position vector $\vec{r} = \langle x, y, z \rangle$ with terminal point on the plane, we have a vector equation of the plane

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

Equivalently,

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Equivalently, we have a **(scalar) equation of the plane**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The scalar equation of the plane can be simplified as **standard form**

$$ax + by + cz + d = 0.$$

Example 7. Find an equation of the plane passing trough the point $(1, 2, 3)$ with normal vector $\langle 4, 5, 6 \rangle$ (or $4\vec{i} + 5\vec{j} + 6\vec{k}$).

Position vector $\vec{r}_0 = \langle 1, 2, 3 \rangle$ and normal vector $\vec{n} = \langle 4, 5, 6 \rangle$
Vector equation is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, so $\langle 4, 5, 6 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$.
Scalar equation is $4(x - 1) + 5(y - 2) + 6(z - 3) = 0$.
Standard equation is $4x + 5y + 6z - 32 = 0$

Definition.

Two planes are **parallel** if their normal vectors are parallel. More generally, the **angle between two planes** is the **acute** angle between the two normal vectors.

Example 8. Find an equation of the plane trough the point $(1, 4, -3)$, and parallel to the plane $5x + y - 2z = 9$.

Normal vector is $\vec{n} = \langle 5, 1, -2 \rangle$
The equation for the plane is $5(x - 1) + (y - 4) - 2(z + 3) = 0$.
Standard form is $5x + y - 2z - 15 = 0$.

Example 9. * Find the angle between the two planes, $x + 2y + z = 5$ and $x + y = 8$.

Two normal vectors $\vec{m} = \langle 1, 2, 1 \rangle$ and $\vec{n} = \langle 1, 1, 0 \rangle$.
$\cos\theta = \dfrac{\vec{m} \cdot \vec{n}}{|\vec{m}||\vec{n}|} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$.
So, $\theta = \arccos \frac{\sqrt{3}}{2} = \pi/6$

Example 10. Find an equation of a line trough the point $(1, 2, 3)$, and perpendicular to the plane $5x + y - 2z = 4$.

Position vector is $\vec{r}_0 = \langle 1, 2, 3 \rangle$.
Direction vector $\vec{v} = \vec{n} = \langle 5, 1, -2 \rangle$
Equation of the line $\vec{r} = \langle 1, 2, 3 \rangle + t\langle 5, 1, -2 \rangle = \langle 1 + 5t, 2 + t, 3 - 2t \rangle$, or symmetric equation

$$\frac{x - 1}{-5} = \frac{y - 2}{1} = \frac{z - 3}{-2}$$

Example 11. Find the intersection point of the line $x = 1 - 2t, y = 3 + t, z = 2 + 4t$ and the plane $2x - y + z = 0$.

Plug in the line into the plane

$$2(1 - 2t) - (3 + t) + (2 + 4t) = 0$$

Solve the equation we have $t = 1$. So $x = -1, y = 4, z = 6$.
The intersection point is $(-1, 4, 6)$.

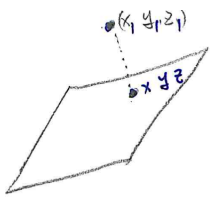
Example 12. Show that the line given by $(x, y, z) = (1, 4, 5) + t(1, 1, 1)$ is parallel to the plane given by $4x - 9y + 5z - 8 = 0$.

Method 1. Check that there is no intersection point by computation as the above example.
Method 2. Check that the direction vector $\vec{v} = \langle 1, 1, 1 \rangle$ of the line is orthogonal to the normal vector $\vec{n} = \langle 4, -9, 5 \rangle$. Compute dot product $\vec{v} \cdot \vec{n} = 0$.

Example 13. Determine whether the planes are parallel, perpendicular, or nether. $x+2y+3z=9$ and $-4x-y+2z=8$.

Normal vectors for two planes are $\vec{m}=\langle 1,2,3\rangle$ and $\vec{n}=\langle -4,-1,2\rangle$.
They are not parallel since they are not scalar multiple.
$\vec{m}\cdot\vec{n}=0$ so $\vec{m}$ and $\vec{n}$ are perpendicular.
So, two planes are perpendicular.

Example 14. *Find a formula for the distance between a point (x_1,y_1,z_1) and a plane $ax+by+cz+d=0$.



The line perpendicular to the plane passing (x_1, y_1, z_1) is

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$$

$$a(x_1+at)+b(y_1+bt)+c(z_1+ct)+d=0$$

$$t=\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}$$

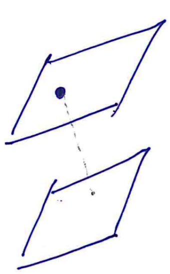
The distance between $(x_1\ y_1\ z_1)$ and $(x\ y\ z)$ is

$$D=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

$$=\sqrt{(at)^2+(bt)^2+(ct)^2}$$

$$=\sqrt{a^2+b^2+c^2}\,|t| = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Example 15. *Calculate the distance between two parallel planes
$2x+3y+z+2=0$ and $4x+6y+2z+1=0$.



$\langle 2,3,1\rangle$ $\quad\langle 4,6,2\rangle = 2\langle 2,3,1\rangle$ are normal vectors.

- Choose a point on the plane $2x+3y+z+2=0$.
  Let $y=z=0 \Rightarrow x=-1$
  $(-1,0,0)$

Using the above example

$$D=\frac{|4(-1)+6(0)+2(0)+1|}{\sqrt{4^2+6^2+2^2}}=\frac{3}{\sqrt{46}}$$