§1.3 Dot product, angles, and orthogonal projection in $\mathbb{R}^n$

Definition.

If $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$ in $\mathbb{R}^n$, then the **dot product** (inner product) of $\vec{a}$ and $\vec{b}$ is a number given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Example 1. $\langle 1, 2, 3 \rangle \cdot \langle -2, 6, 2/3 \rangle = -2 + 12 + 2 = 12.$

Theorem.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then

(1) $\vec{a} \cdot \vec{a} = |a|^2$

(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(4) $\vec{0} \cdot \vec{a} = 0$

(5) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

Let us verify (1) and (2) in $\mathbb{R}^3$, the others are similarly.

- $\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 = |a|^2$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$

$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Theorem. (Cauchy-Schwarz Inequality)

Let $\vec{a}$ an $\vec{b}$ be vectors in $\mathbb{R}^n$. Then,

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|.$$

In particular, $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|$ if and only if $\vec{a}$ and $\vec{b}$ are parallel.

Hint for proof: Consider the the fact $|\vec{a} + t\vec{b}|^2 \geq 0$ for any real number t.

Theorem. Triangle Inequality

Let $\vec{a}$ an $\vec{b}$ be vectors in $\mathbb{R}^n$. Then,

$$|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|.$$

In particular, $|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|$ if and only if a and b have the same direction.

Hint for proof: consider the square of both sides and then use Cauchy-Schwarz inequality.

If θ ($0 \leq \theta \leq \pi$) is the **angle** between the vector $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

Or, we can write the equality as

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|} \text{ or } \theta = \arccos\frac{\vec{a}\cdot\vec{b}}{|a||b|}$$

In particular, if $\vec{a}$ and $\vec{b}$ are parallel, then $\theta = 0$ or π.

Remark: In $\mathbb{R}^2$ and $\mathbb{R}^3$, the angle is already defined in trigonometry, the above formula is a theorem proved with the help of *Law of Cosines* from trigonometry:

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

For $\mathbb{R}^n$, $n \geq 4$, we use it as a definition for the angle between two vectors.

Example 2. If the vectors $\vec{a}$ and $\vec{b}$ have length 4 and 6, and the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, find $\vec{a}\cdot\vec{b}$.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\frac{\pi}{6}) = 4(6)(\sqrt{3}/2) = 12\sqrt{3}$.

Example 3. Find the angle between $\vec{a} = \langle 1,2,3\rangle$ and $\vec{b} = \langle 2,-1,2\rangle$.

First, calculate $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 6$.

So, the angle $\theta = \arccos\dfrac{\vec{a}\cdot\vec{b}}{|a||b|} = \arccos(2/\sqrt{14}) \approx 1.01 \approx 57.69°$

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal**, if the angle between them is $\theta = \pi/2$.

Theorem.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$.

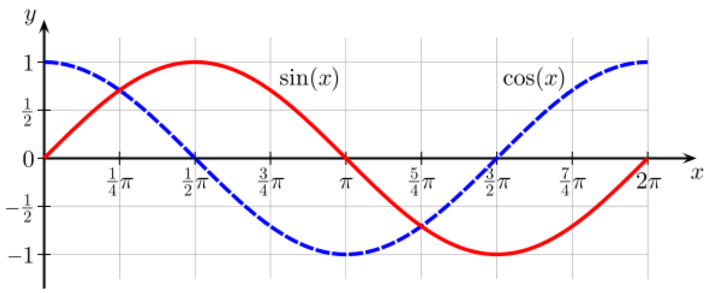
Example 4. Show that $3\vec{i} + 2\vec{j} - \vec{k}$ is perpendicular to $3\vec{i} - 5\vec{j} - \vec{k}$.

$(3\vec{i} + 2\vec{j} - \vec{k}) \cdot (3\vec{i} - 5\vec{j} - \vec{k}) = 0.$

Theorem.

If θ is **acute** $(0 \le \theta < \pi/2)$, then $\cos\theta > 0$. Thus $\vec{a} \cdot \vec{b} > 0$.
If θ is **obtuse** $(\pi/2 \le \theta \le \pi)$, then $\cos\theta < 0$. Thus $\vec{a} \cdot \vec{b} < 0$.
When $\vec{a}$ and $\vec{b}$ are in the **same direction** $(\theta = 0)$, then $\cos\theta = 1$. Thus, $\vec{a} \cdot \vec{b} = |a||b|$.
When $\vec{a}$ and $\vec{b}$ are in the **opposite direction** $(\theta = \pi)$, then $\cos\theta = -1$. Thus, $\vec{a} \cdot \vec{b} = -|a||b|$.

The theorem follows from $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.



Example 5. Find a unit vector which is orthogonal to both $\langle 1, 1, 0\rangle$ and $\langle 1, 2, 3\rangle$.

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$ be such vector.

Then $\langle a_1, a_2, a_3\rangle \cdot \langle 1, 1, 0\rangle = a_1 + a_2 = 0$

$\langle a_1, a_2, a_3\rangle \cdot \langle 1, 2, 3\rangle = a_1 + 2a_2 + 3a_3 = 0$

Next, we need to solve $\begin{cases} a_1 + a_2 = 0 \\ a_1 + 2a_2 + 3a_3 = 0 \end{cases}$

$a_1 = -a_2$

$-a_2 + 2a_2 + 3a_3 = 0 \Rightarrow a_2 = -3a_3$

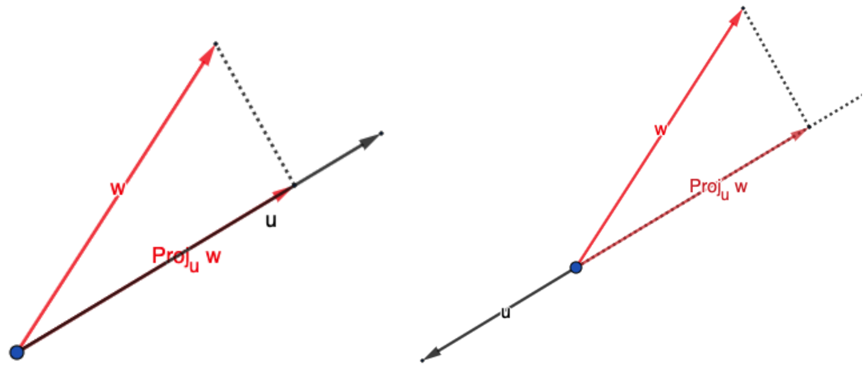
One solution will be $\begin{cases} a_1 = 3 \\ a_2 = -3 \\ a_3 = 1 \end{cases}$ $\vec{b} = \langle 3, -3, 1\rangle$

$|\vec{b}| = \sqrt{9+9+1} = \sqrt{19}$

So $\vec{a} = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{19}}\langle 3, -3, 1\rangle = \langle \frac{3}{\sqrt{19}}, -\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\rangle$

▶ Orthogonal Projections in $\mathbb{R}^n$

The **orthogonal projection of $\vec{w}$ onto $\vec{u}$** is denoted by $\text{proj}_{\vec{w}}\vec{u}$. It is also called the **component of $\vec{w}$ parallel to $\vec{u}$.**



The **component of $\vec{w}$ normal (or orthogonal) to $\vec{u}$** is the vector

$$\vec{w} - \text{proj}_{\vec{w}}\, \vec{u}$$

Formula for projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\, \vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$$

Proof: Suppose $\text{proj}_{\vec{a}}\, \vec{b} = x\vec{a}$. Then, $\vec{a}\cdot(\vec{b} - x\vec{a}) = 0$. That is $\vec{a}\cdot\vec{b} - \vec{a}\cdot x\vec{a} = 0$. Hence $\vec{a}\cdot\vec{b} = x(\vec{a}\cdot\vec{a})$ and so $x = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}$.

Example 6. Find the orthogonal projection of $\vec{b} = \langle 7, 6, 3\rangle$ onto $\vec{a} = \langle 4, 2, 0\rangle$.

$$\text{proj}_{\vec{a}}\, \vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \left(\frac{40}{20}\right)\langle 4, 2, 0\rangle = \langle 8, 4, 0\rangle$$

Example 7. Find the components of $\vec{F} = \langle 1, 2, 6\rangle$ parallel and normal to $\vec{v} = \langle 1, 1, 2\rangle$.

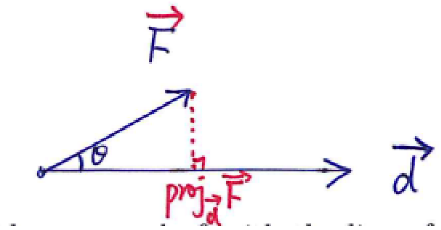
The components of $\vec{F}$ parallel to $\vec{v}$ is
$\text{proj}_{\vec{v}}(\vec{F}) = \left(\dfrac{\vec{F}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \dfrac{15}{6}\langle 1, 1, 2\rangle = \langle 5/2, 5/2, 5\rangle$
The components of $\vec{F}$ normal to $\vec{v}$ is
$\vec{F}_n = \vec{F} - \text{proj}_{\vec{v}}(\vec{F}) = \langle -3/2, -1/2, 1\rangle$

One application of projection in physics is calculating the **work** W by a constant force F moving an object through a distance d. If the force F is along the line of motion, then the work $W = Fd$.

Definition.

If F in $\mathbb{R}^n$ has an angle θ with the line of motion, then the **work** is defined by $W = (|\vec{F}|\cos\theta)|\vec{d}|$. Moreover

$$W = |\vec{F}||\vec{d}|\cos\theta = \vec{F}\cdot\vec{d}.$$



Example 8. A cart is pulled 100 meters along a horizontal path with a force of 60 N exerted at an angle of 25° above the horizontal. Find the work done by the force.

The work done by the force is

$$W = \vec{F}\cdot\vec{d} = |\vec{F}||\vec{d}|\cos\theta = 60(100)\cos 25^\circ \approx 5437\ N\cdot m$$

Example 9. A force is given by a vector $\vec{F} = \langle -1, 2, 4\rangle$ and move a particle from the point $A(2,2,1)$ to the point $B(0,3,2)$. Find the work done by the force.

The displacement vector $\vec{d} = \overrightarrow{AB} = B - A = \langle -2, 1, 1\rangle$.
The work is $W = \vec{F}\cdot\vec{d} = 2 + 2 + 4 = 8$.