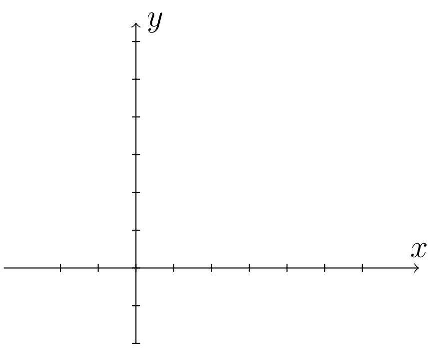
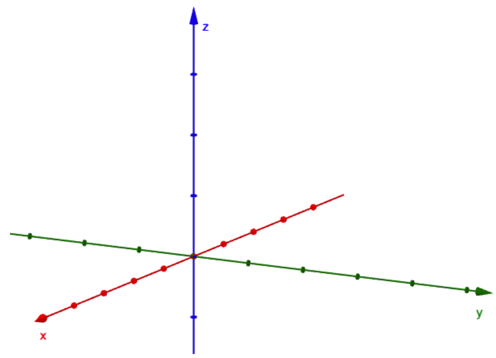
§1.1 Euclidean space $\mathbb{R}^n$

2D space $\mathbb{R}^2$. 2-dimensional **coordinate system** for a plane $\mathbb{R}^2$ using **coordinate axes**, labeled by the x-axis and y-axis. A place $\mathbb{R}^2$ is divided by **4 quadrants**. The arrow direction is the **positive** direction. A **point** in $\mathbb{R}^2$ is described by an order pair (x, y).

3D space $\mathbb{R}^3$. The 3-dimensional **coordinate system** for $\mathbb{R}^3$ include coordinate axes, labeled by the x-axis, y-axis, and z-axis following the **right-hand rule**. $\mathbb{R}^3$ is divided by 8 **octants**. The 3-dimensional space $\mathbb{R}^3$ can be written as Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\})$. A point in $\mathbb{R}^3$ is described by an order pair (x, y, z).

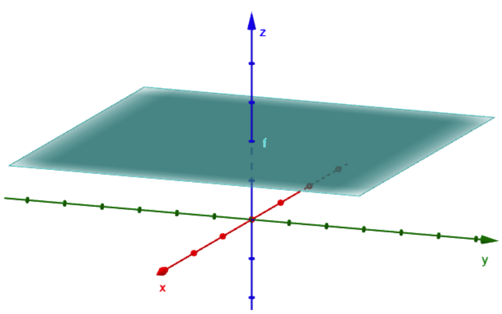




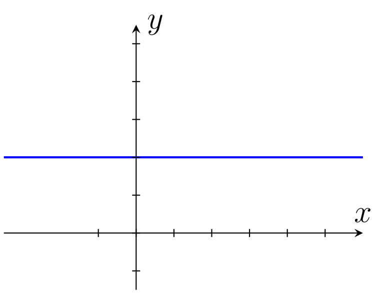
The **Projection** of a point P (a, b, c) onto the xy-plane is $(a, b, 0)$, i.e. we drop perpendicular from P to the xy-plane.
The Projection of a point (a, b, c) onto the yz-plane is $(0, b, c)$.
The Projection of a point (a, b, c) onto the xz-plane is $(a, 0, c)$.

An equation in the variables x and y is a **curve** in $\mathbb{R}^2$.
An equation in the variables x, y and z is a **surface** in $\mathbb{R}^3$.

Example 1. $z = 2$ in $\mathbb{R}^3$. (A plane parallel to the xy-plane.)



Example 2. $y = 2$ in $\mathbb{R}^2$. (A line parallel to the x-axis.)



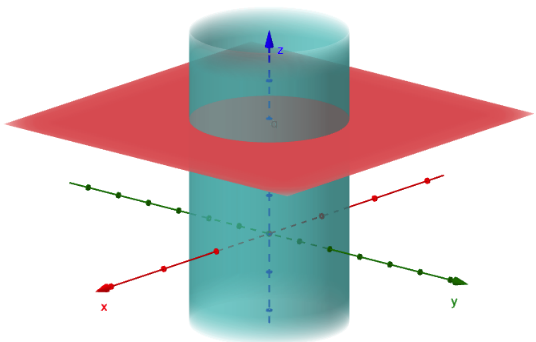
Use https://www.geogebra.org/3d to look at the following examples.

Example 3. $y = 2$ in $\mathbb{R}^3$. (A plane parallel to the xz-plane.)

Example 4. Which points (x, y, z) in $\mathbb{R}^3$ satisfy $x^2 + y^2 = 4$.

Example 5. Describe and sketch the surface by $x = y$ in $\mathbb{R}^3$.

Example 6. Which points (x, y, z) in $\mathbb{R}^3$ satisfy $x^2 + y^2 = 4$ and $z = 3$.



Definition. Distance Formula in $\mathbb{R}^n$

The **distance** $|AB|$ between the points A $(a_1, a_2, ..., a_n)$ and B $(b_1, b_2, ..., b_n)$ in $\mathbb{R}^n$ is

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

In particular, the **distance** $|P_1P_2|$ between the points P_1 (x_1, y_1, z_1) and P_2 $(x_2, y_2, z_2) \in R^3$ is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Example 7. Find the distance from P $(1, 2, 3)$ to Q $(3, 4, 2)$.

$$|PQ| = \sqrt{(1-3)^2 + (2-4)^2 + (3-2)^2} = \sqrt{4+4+1} = 3.$$

Example 8. (Equation of a Sphere) Equation for a sphere of radius r and center (h, k, l) is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

In particular, if the center is the origin $(0, 0, 0)$, then the equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Proof of the distance formula in $\mathbb{R}^3$:
P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are two points in $\mathbb{R}^3$
Let $A = (x_2, y_1, z_2)$ and $B(x_2, y_2, z_1)$. By Pythagorean Theorem, $|P_1B|^2 = |P_1A|^2 + |AB|^2$ and

$$\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\ &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \end{aligned}$$

Example 9. Determine whether the points A $(2, 4, 2)$, B $(3, 7, -2)$, and C $(1, 3, 3)$ lie on a straight line.

By calculation, $|AB| = \sqrt{26}$, $|AC| = \sqrt{3}$, $|BC| = \sqrt{45}$. So, sum of 2 line segments does not equal to the other one. Hence, they are not on a straight line.

Example 10. Find the equation of a sphere with center $(2, -5, 1)$ and radius 2. How does this sphere intersect the 3 coordinate planes?

The equation of a sphere with center $(2, -5, 1)$ and radius 2 is

$$(x - 2)^2 + (y - 5)^2 + (z - 1)^2 = 4$$

1. Intersect with xy-plane.
The equation for xy-plane is $z = 0$.
The intersection is given by $(x-2)^2+(y-5)^2 = 3$, which is a circle on xy-plane of radius $\sqrt{3}$.
2. Intersect with xz-plane.
No intersection.
3. Intersect with yz-plane.
A single point $(-5, 1)$.

Example 11. Find the equation of a sphere if one of its diameters has endpoints $A(2, -1, 1)$ and $B(-2, -3, 2)$.

The diameter $d = |AB| = \sqrt{21}$, so the radius $r = \sqrt{21}/2$.
Center point of the sphere is the middle of AB given by $(0, -2, 3/2)$
Equation of the sphere is $x^2 + (y + 2)^2 + (z - 3/2)^2 = 21/4$.

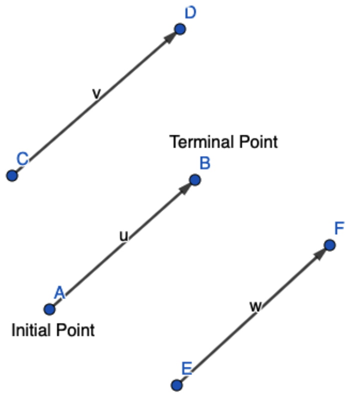
Example 12. Describe the region represented by the inequality $x^2 + y^2 + z^2 > 2z$.

$x^2 + y^2 + z^2 > 2z$ is equivalent to $x^2 + y^2 + z^2 - 2z > 0$, which is equivalent to $x^2 + y^2 + (z-1)^2 - 1 > 0$ or $x^2 + y^2 + (z-1)^2 > 1$
The region is $\mathbb{R}^3$ with a solid ball of radius 1 with center $(0, 0, 1)$ removed.

§1.2 $\mathbb{R}^n$ as a vector space

A **vector** in $\mathbb{R}^n$ is a quantity that has magnitude and direction. A vector is often represented by a directed line segment, denoted by $\mathbf{u}$ or $\vec{u}$. (A detailed study of vector space is in class Math-2331 Linear Algebra)

The **displacement vector** $\overrightarrow{AB}$ a directed line segment from **initial point** A to **terminal point** B.



The displacement vector $\overrightarrow{AB}$ has the same magnitude(length) and the same direction as $\overrightarrow{CD}$ (and $\overrightarrow{EF}$) even though the are in a different position.

We say that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are **equivalent** (or equal) and we write $\overrightarrow{AB} = \overrightarrow{CD}$.

Zero vector has no direction, denoted by $\vec{0}$.

A vector $\vec{v}$ starting from **origin** to a point P ((a_1, a_2) or (a_1, a_2, a_3), depending on $\mathbb{R}^2$ or $\mathbb{R}^3$) is called the **position vector** of P. The coordinates are called the **components** of $\vec{v}$. We denote $\vec{v} = \langle a_1, a_2, a_3\rangle$ or $\vec{v} = (a_1, a_2, a_3)$ as in the book.

Given the points $A(x_1, x_2, x_3)$ and $B(x_2, y_2, z_2)$, the vector $\vec{v}$ with representation $\overrightarrow{AB}$ is

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle.$$

Example 1. Find the displacement vector represented by the directed line segment with initial point $A(1, -2, 3)$ and terminal point $B(2, 1, 5)$.

Answer: $\overrightarrow{AB} = \langle 1, 3, 2\rangle$.

The **magnitude** or **length** of the vector $v = \langle a, b, c\rangle$ in $\mathbb{R}^3$ is

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}.$$

The length of $v = \langle a, b\rangle$ in $\mathbb{R}^2$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

The magnitude of the velocity vector $\vec{v}$ is called **speed**.

Example 2. The speed of a velocity vector $\vec{v} = \langle 3, 4\rangle$ m/s is $|\vec{v}| = 5$ m/s.

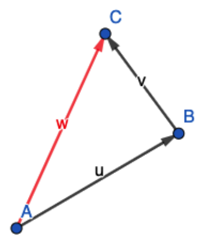
Example 3. Consider the force vector $\vec{F} = \langle 1, 3, 2\rangle$ Newtons. Find the magnitude of $\vec{F}$.

The magnitude of $\vec{F}$ is $|\vec{F}| = \sqrt{1+9+4} = \sqrt{14}$ Newtons

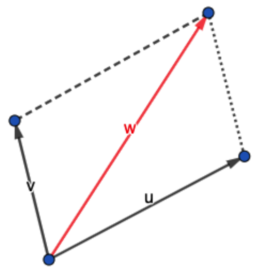
▶ Operations of vectors

1. Sum of two vectors

The Triangle Law for $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ and the Parallelogram Law for $\vec{w} = \vec{u} + \vec{v}$.



or



If $\vec{u} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\vec{v} = \langle b_1, b_2, \ldots, b_n\rangle$, then the **sum** is coordinate-wise sum

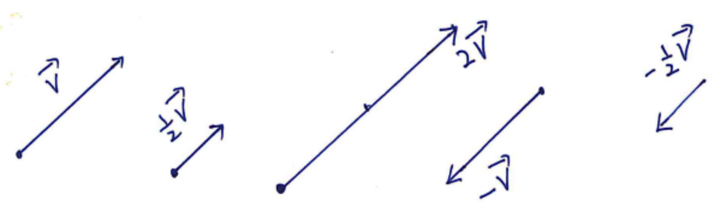
$$\vec{u} + \vec{v} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\rangle$$

2. Scalar multiplication

If $c \in \mathbb{R}$ is a scalar and $\vec{v}$ is a vector, then the **scalar multiplication** $c\vec{v}$ is the vector whose length $|c|$ is times the length of $\vec{v}$.
The direction of $c\vec{v}$ is the same as $\vec{v}$ if $c > 0$.
The direction of $c\vec{v}$ is opposite to $\vec{v}$ if $c < 0$.
If $c = 0$ or $\vec{v} = \vec{0}$, then $c\vec{v} = \vec{0}$.

If $\vec{u} = \langle a_1, a_2, \ldots, a_n\rangle$ and $k \in \mathbb{R}$, then the **scalar multiplication** is

$$k\vec{u} = \langle ka_1, ka_2, \ldots, ka_n\rangle$$



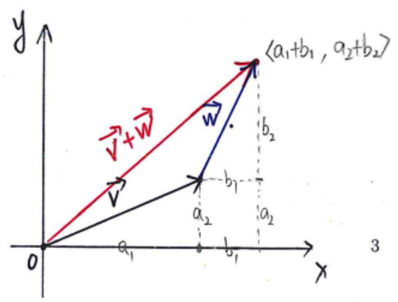
Theorem.

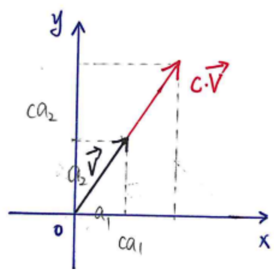
Two vectors $\vec{v}$ and $\vec{w}$ have the the *same* direction if and only if $\vec{v} = k\vec{w}$ for some $k > 0$.
Two vectors $\vec{v}$ and $\vec{w}$ have the *opposite* direction if and only if $\vec{v} = c\vec{w}$ for some $c < 0$.

3. The **difference** of two vectors can be defined using sum and scalar product:

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$$

Example 4. If $\vec{v} = \langle a_1, a_2 \rangle$ and $\vec{w} = \langle b_1, b_2 \rangle$, then
$\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2 \rangle$; $\vec{v} - \vec{w} = \langle a_1 - b_1, a_2 - b_2 \rangle$; and $c\vec{v} = \langle ca_1, ca_2 \rangle$.





The magnitude of $\vec{a} - \vec{b}$ is called the distance of $\vec{a}$ and $\vec{b}$.

Example 5. If $\vec{a} = \langle 2, 3, 0 \rangle$ and $\vec{b} = \langle -1, 2, 4 \rangle$, find $|\vec{a}|$, $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, and $2\vec{a} + 3\vec{b}$.

$|\vec{a}| = \sqrt{13}$; $\vec{a} + \vec{b} = \langle 1, 5, 4 \rangle$; $\vec{a} - \vec{b} = \langle 3, 1, -4 \rangle$; $3\vec{b} = \langle -3, 6, 12 \rangle$; $2\vec{a} + 3\vec{b} = \langle 1, 12, 12 \rangle$.

Theorem. Algebraic Properties

For $\vec{u}$ and $\vec{w}$ vectors in $\mathbb{R}^n$, and c, d scalars, the following algebraic properties hold.

(1) $\vec{u} + \vec{w} = \vec{w} + \vec{u}$
(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$
(3) $\vec{u} + \vec{0} = \vec{u}$
(4) $\vec{u} + (-\vec{u}) = \vec{0}$
(5) $c(\vec{u} + \vec{w}) = c\vec{u} + c\vec{w}$
(6) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
(7) $c(d\vec{u}) = (cd)\vec{u}$
(8) $1\vec{u} = \vec{u}$

Theorem.

$$|k\vec{v}| = |k| \cdot |\vec{v}|$$

Example 6. Suppose $|\vec{v}| = 3$, what is the magnitude of $-2\vec{v}$?

$|-2\vec{v}| = 2|\vec{v}| = 6$

Standard basis vectors in $\mathbb{R}^3$: $\vec{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \langle 0, 1, 0\rangle$, $\vec{k} = \langle 0, 0, 1\rangle$.
We can express any vector $\vec{v} = \langle a, b, c\rangle$ as a **linear combination** of $\vec{i}$, $\vec{j}$ and $\vec{k}$, as

$$\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}.$$

Example 7. If $\vec{a} = 2\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{b} = 3\vec{i} - 6\vec{j}$, express the vector $3\vec{a} - 2\vec{b}$ in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.

$$\begin{aligned} 3\vec{a} - 2\vec{b} &= 3(2\vec{i} + \vec{j} - 4\vec{k}) - 2(3\vec{i} - 6\vec{j}) \\ &= 6\vec{i} + 3\vec{j} - 2\vec{k} - 6\vec{i} + 12\vec{j} \\ &= 15\vec{j} - 12\vec{k} \end{aligned}$$

Definition.

A **unit vector** is a vector whose length is 1. For example, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors. In general, if $\vec{a} \neq \vec{0}$, then the unit vector that has the same direction as $\vec{a}$ is

$$\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \frac{\vec{a}}{|\vec{a}|}.$$

It is called the **direction** of $\vec{a}$.

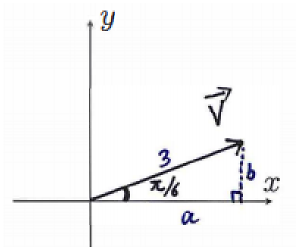
Example 8. Find the unit vector in the direction of $\vec{a} = -2\vec{i} - 3\vec{j} + \vec{k}$.

$|\vec{a}| = \sqrt{4 + 9 + 1} = \sqrt{14}$.
So, the unit vector is $\dfrac{\vec{a}}{\sqrt{14}} = \dfrac{-2\vec{i} - 3\vec{j} + \vec{k}}{\sqrt{14}} = -\frac{2}{\sqrt{14}}\vec{i} - \frac{3}{\sqrt{14}}\vec{j} + \frac{1}{\sqrt{14}}\vec{k}$

Example 9. Find the vector $\vec{u}$ in the same direction of $\vec{a} = 1\vec{i} - 2\vec{j} + \vec{k}$ with magnitude 5.

$\vec{u} = 5(\vec{a}/|\vec{a}|) = 5(1\vec{i} - 2\vec{j} + \vec{k})/\sqrt{6}$

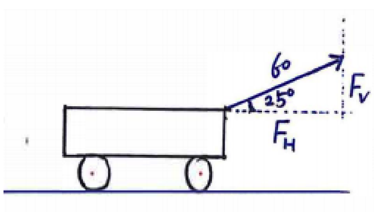
Example 10. If $\vec{v}$ lies in the first quadrant and make an angle of $\pi/6$ with the positive x-axis and $|\vec{v}| = 3$, find $\vec{v}$ in component form.



$\vec{a} = 3\cos(\frac{\pi}{6}) = 3(\frac{\sqrt{3}}{2})$ and $\vec{b} = 3\sin(\frac{\pi}{6}) = 3(\frac{1}{2})$. Thus, $\vec{v} = \langle \frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle$

Example 11. A cart is pulled along a horizontal path with a forth of 60 N exerted at an angle of 25° above the horizontal.

Find the horizontal and vertical components of the force.



$\vec{F}_H = 60\cos 25^\circ \approx 54.4N$ and $\vec{F}_v = 60\sin 25^\circ \approx 25.4N$

Theorem. Newton's 2nd Law of Motion

Let $\vec{F}$ is the net(total) force vector acting on an object, m is the objects mass, and $\vec{a}$ is the acceleration of the object. Newton's 2nd Law of Motion is given by the scalar product:

$$\vec{F} = m\vec{a}$$

Example 12. If $F_1 = 3\vec{i} - 3\vec{j} + \vec{k}$, $F_2 = \vec{i} - 4\vec{j} + 3\vec{k}$, and $F_3 = -2\vec{i} + 5\vec{j}$ acts on an object with mass 2 kilograms, determine the acceleration a of the object, and the magnitude of the acceleration.

The net force $F = F_1 + F_2 + F_3 = 2\vec{i} - 2\vec{j} + 4\vec{k}$. By Newton's 2nd Law of Motion, the acceleration is $\vec{a} = \frac{1}{2}\vec{F} = \langle 1, -1, 2 \rangle \ m/s^2$.
The magnitude of the acceleration is $|\vec{a}| = \sqrt{6} \ m/s^2$.

Newtons Law of Universal Gravitation: A mass M exerts a gravitational attraction force on the other mass m,

$$F_{M \leftarrow m} = \frac{GMm}{|\vec{r}|^2}\left(\frac{\vec{r}}{|\vec{r}|}\right)$$

Here $\vec{r}$ is the displacement vector $\vec{r} = P_{Mm} = P_M - P_m$. G is universal gravitational constant, $G = 6.67 \times 10^{-11} \ m^3/(kg \cdot s^2)$

The force $F_{m \leftarrow M}$ that m exerts on M is $F_{m \leftarrow M} = -F_{M \leftarrow m}$