Review the composition of two functions.
If we have two functions $f(x)$ and $g(x)$, we can construction a new function $f(g(x))$, which is called the **composition** of $f(x)$ and $g(x)$. Here, $f(x)$ is called the **outside function**, $g(x)$ is called the **inside function**.

Example: Let $f(u) = 2\ln u$ and $u(x) = 4x^2 + 3$. Then the composition function

$$f(u(x)) = 2\ln(3x^2 + 3).$$

Example: Let $f(u) = \sqrt{u}$ and $u(x) = 2x^3 + x$. Then the composition function

$$f(u(x)) = \sqrt{2x^3 + x}.$$

Question: How to compute the derivative of $h(x) = 2\ln(3x^2 + 3)$?

Chain Rule for Derivative.

> If $h(x)$ is a composition function $f(u(x))$, then
>
> $$h'(x) = f'(u) \cdot u'(x)$$

In the formula $f'(u)$, u is a shorthand for $u(x)$.

Example: Compute the derivative of $h(x) = 2\ln(3x^2 + 3)$

$u(x) = 3x^2+3$ $\quad u'(x) = 6x$ $\quad h'(x) = f'(u)\cdot u'(x)$

$f(u) = 2\ln u$ $\quad f'(u) = \frac{2}{u}$ $\quad = \frac{2}{u}\cdot 6x = \frac{2}{3x^2+3}\cdot 6x$

Example: Compute the derivative of $h(x) = \sqrt{2x^3 + x}$

$u(x) = 2x^3+x$ $\quad u'(x) = 6x^2+1$ $\quad h'(x) = f'(u)\cdot u'(x)$

$f(u) = \sqrt{u} = u^{\frac{1}{2}}$ $\quad f'(u) = \frac{1}{2}u^{-\frac{1}{2}}$ $\quad = \frac{1}{2}u^{-\frac{1}{2}}(6x^2+1)$

$= \frac{1}{2}(2x^3+x)^{-\frac{1}{2}}(6x^2+1)$

Example: Compute the derivative of $f(x) = 3e^{\sqrt{x}+3x^2}$

$u(x) = \sqrt{x}+3x^2 = x^{\frac{1}{2}}+3x^2$ $\quad u'(x) = \frac{1}{2}x^{-\frac{1}{2}}+6x$

$g(u) = 3e^u$ $\quad g'(u) = 3e^u$

$f'(x) = g'(u)\cdot u'(x) = 3e^u\cdot\left(\frac{1}{2}\cancel{x}^{-\frac{1}{2}}+6x\right) = 3e^{x^{\frac{1}{2}}+3x^2}\cdot\left(\frac{1}{2}x^{-\frac{1}{2}}+6x\right)$

Example: Compute the derivative of $f(x) = 3(\sqrt[3]{x} + 3x)^8$

$u(x) = \sqrt[3]{x} + 3x = x^{\frac{1}{3}} + 3x$ $\quad u'(x) = \frac{1}{3}x^{-\frac{2}{3}} + 3$

$h(u) = 3u^8$ $\quad h'(u) = 24u^7$

$f'(x) = h'(u)u'(x) = 24u^7 \cdot (\frac{1}{3}x^{-\frac{2}{3}} + 3) = 24(x^{\frac{1}{3}} + 3x)^7(\frac{1}{3}x^{-\frac{2}{3}} + 3)$

Example: Compute the derivative of $h(x) = 15\ln(3e^x + x^3)$

$u(x) = 3e^x + x^3$ $\quad u'(x) = 3e^x + 3x^2$

$f(u) = 15\ln u$ $\quad f'(u) = \frac{15}{u}$

$h'(x) = f'(u)u'(x) = \frac{15}{u}(3e^x + 3x^2) = \frac{15}{3e^x + x^3}(3e^x + 3x^2)$

Example: Compute the derivative of $g(x) = \dfrac{45}{3\ln x + 3^x} + e$

$u(x) = 3\ln x + 3^x$ $\quad u'(x) = \frac{3}{x} + (\ln 3)3^x$

$f(u) = 45u^{-1} + e$ $\quad f'(u) = -45u^{-2}$

$g'(x) = -45u^{-2}(\frac{3}{x} + (\ln 3)3^x) = -45(3\ln x + 3^x)^{-2}(\frac{3}{x} + (\ln 3)3^x)$

Example: Compute the derivative of $f(x) = \dfrac{3}{(4x^5 - \sqrt[3]{x^4})^4} + x^5$

$u(x) = 4x^5 - \sqrt[3]{x^4} = 4x^5 - x^{\frac{4}{3}}$ $\quad u'(x) = 20x^4 - \frac{4}{3}x^{\frac{1}{3}}$

$g(u) = \frac{3}{u^4} = 3u^{-4}$ $\quad g'(u) = -12u^{-5}$

$f'(x) = g'(u)u'(x) + 5x^4$

$= -12(4x^5 - x^{\frac{4}{3}})^{-5}(20x^4 - \frac{4}{3}x^{\frac{1}{3}}) + 5x^4$

Example: Compute the derivative of $h(x) = 2e^{\sqrt[5]{x} + 3e^x} + \ln x$

$u(x) = \sqrt[5]{x} + 3e^x = x^{\frac{1}{5}} + 3e^x$ $\quad u'(x) = \frac{1}{5}x^{-\frac{4}{5}} + 3e^x$

$f(u) = 2e^u$ $\quad f'(u) = 2e^u$

$h'(x) = f'(u)u'(x) + \frac{1}{x} = 2e^{x^{\frac{1}{5}} + 3e^x}(\frac{1}{5}x^{-\frac{4}{5}} + 3e^x) + \frac{1}{x}$

Exercise Compute the derivative of the following functions

1. $f(x) = \dfrac{e}{(4x^2 - \sqrt[3]{x^2})^2}$

$u(x) = 4x^2 - x^{\frac{2}{3}}$ $\quad u'(x) = 8x - \frac{2}{3}x^{-\frac{1}{3}}$

$g(u) = e \cdot u^{-2}$ $\quad g'(u) = -2e\,u^{-3}$

$f'(x) = g'(u)u'(x) = -2e(4x^2 - x^{\frac{2}{3}})^{-3}(8x - \frac{2}{3}x^{-\frac{1}{3}})$

2. $f(x) = \ln(5x^4 + 5x) + e^{2x}$

$f'(x) = \frac{20x^3 + 5}{5x^4 + 5x} + 2e^{2x}$

3. $f(x) = \sqrt[3]{e^x + 3\ln x + e}$

$f'(x) = \frac{1}{3}(e^x + 3\ln x + e)^{-\frac{2}{3}}(e^x + \frac{3}{x})$