1. The definition of Limits

- The limits of $f(x)$ when x approach a is denoted by

$$\lim_{x\to a} f(x)$$

- $x \to a$ means that x approach a, i.e. x is close to a very much. $(x \neq a)$

Example 1:

$$\lim_{x\to 2}(x^2+1) = 5.$$

Example 2:

$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1}\left(\frac{(x+1)(x-1)}{x-1}\right) = \lim_{x\to 1}(x+1) = 2.$$

When the function $f(x)$ is continuous and $f(a)$ is well-defined, then

$$\lim_{x\to a} f(x) = f(a)$$

denominator is NOT zero

2. Derivative at a point by Limits

$\boxed{\text{Rate of change at } x=a}$ = $\boxed{\text{slope of the tangent line}}$

The derivative of $f(x)$ at a is given by $$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Example 3:(Q1 is Homework2 in textbook section 2.4 page 163)
Q1, Estimate the derivative of function $f(x) = -x^2 + 4x$ at $x = 3$, to the nearest tenth. Calculator work in the last page.
Q2, Using limit definition compute $f'(3)$.

Q1

$x \to 3$	$f(x)$	slope of secant line $\frac{f(x)-f(3)}{x-3}$
2.99	3.019	-1.99
2.999	3.001999	-1.999
3.01	2.9799	-2.01
3.001	2.997999	-2.001

$f'(3) \approx -2.0$

Q2

$$f'(3) = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3}\frac{-x^2+4x-(-3^2+4\times 3)}{x-3}$$

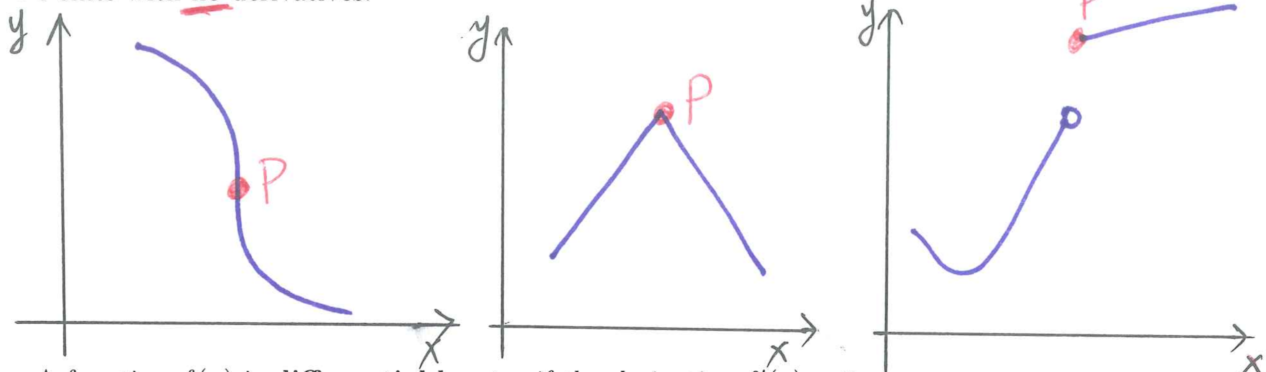
$$= \lim_{x\to 3}\frac{-x^2+4x-3}{x-3}$$

$$= \lim_{x\to 3}\frac{-(x-1)(x-3)}{x-3}$$

$$= \lim_{x\to 3} -(x-1) = -(3-1) = -2$$

3. Differentiablility of a function

• Points with no derivatives.



• A function $f(x)$ is **differentiable at** a if the derivative $f'(a)$ exits.

• A function $f(x)$ is **differentiable** over an open interval if the derivative $f'(x)$ exits for all points in the interval.

4. Derivative of a function by Limits

★ **The derivative of function $f(x)$ is given by**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{\partial f(x)}{\partial x} := f'(x)$ $\quad f'(a) = \frac{\partial f(x)}{\partial x}\Big|_{x=a}$

Example 4: Let $f(x) = 2x^2$.

(a) Find the average rate of change of $f(x)$ between the points $(x, f(x))$ and $(x+h, f(x+h))$

$$ARC = \frac{f(x+h) - f(x)}{x+h-x} = \frac{2(x+h)^2 - 2x^2}{h} = \frac{2(x^2+2xh+h^2) - 2x^2}{h}$$

$$= \frac{2x^2 + 4xh + 4h^2 - 2x^2}{h} = \frac{4xh + 4h^2}{h} = 4x + 4h$$

(b) Find $f'(x)$ using the limit definition and the answer (a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}(4x+4h) = 4x$$

Example 5: Let $f(x) = 1 - 2x - 3x^2$.
(a) Find the average rate of change of $f(x)$ between the points $(x, f(x))$ and $(x+h, f(x+h))$

$$ARC = \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{1-2(x+h)-3(x+h)^2-(1-2x-3x^2)}{h}$$

$$= \frac{1-2x-2h-3x^2-6xh-3h^2-1+2x+3x^2}{h}$$

$$= \frac{-2h-6xh-3h^2}{h}$$

$$= -2-6x-3h$$

(b) Find $f'(x)$ using the limit definition and the answer (a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(-2-6x-3h) = -2-6x$$

Example 6: Let $f(x) = 5 - 3x + 4x^2$.
(a) Find the average rate of change of $f(x)$ between the points $(x, f(x))$ and $(x+h, f(x+h))$

$$ARC = \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{5-3(x+h)+4(x+h)^2-(5-3x+4x^2)}{h}$$

$$= \frac{5-3x-3h+4x^2+8xh+4h^2-5+3x-4x^2}{h}$$

$$= \frac{-3h+8xh+4h^2}{h}$$

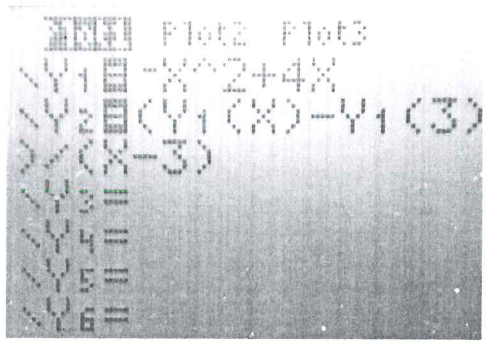
$$= -3+8x+4h$$

(b) Find $f'(x)$ using the limit definition and the answer (a)

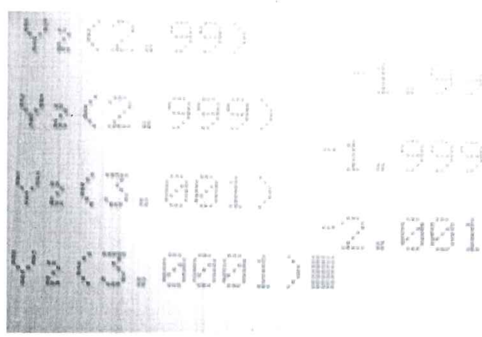
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(-3+8x+4h) = -3+8x$$

Calculator work for Example Q1: (HW2 in textbook section 2.4, page163)
Estimate the derivative of function $f(x) = -x^2 + 4x$ at $x = 3$, to the nearest tenth.

•1. Press "Y=".
•2. After Y1=, put in $-X^2 + 4X$
•3. After Y2=, put in $(Y1(X) - Y1(3))/(X - 3)$
Press (-) for the negative sign.
Press "X,T,θ,n" for X.
Press VARS, ▶, **Enter**, **Enter** for Y1.



•4. Go back to Screen. Press "2ND" then Press "MODE"(QUIT).
•5. Put in Y2(2.99), then **Enter**.
Press VARS, ▶, scroll down, then **Enter**, **Enter** for Y2.



Exercise: Let $f(x) = -x^2 + 4x$.
(a) Find the average rate of change of $f(x)$ between the points $(x, f(x))$ and $(x+h, f(x+h))$
(b) Find $f'(x)$ using the limit definition and the answer (a)
(c) Verify the result above

$$ARC = \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \frac{-(x+h)^2 + 4(x+h) - (-x^2+4x)}{h}$$

$$= \frac{-x^2 - 2xh - h^2 + 4x + 4h + x^2 - 4x}{h}$$

$$= \frac{-2xh - h^2 + 4h}{h} = -2x - h + 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} (-2x - h + 4)$$

$$= -2x + 4$$