GRADUATE STUDENT SEMINAR TALK: (2015 Feb.)

Some functors from topological spaces to lie algebras and algebraic varieties

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ABSTRACT. A common theme in algebraic topology is to study functors from the homotopy category of topological spaces to various categories of algebraic objects, such as the classical homology functor, the cohomology functor and the homopoty functor. In this talk, I will introduce some functors from topological spaces to Lie algebras and algebraic varieties, and explore their properties. We can use these functors study the properties of the original topological spaces. We will also use these functors to study the finitely generated groups by studying the corresponding Eilenberg-MaClane spaces. At last, I will state some results from my joint work with Professor Alexander Suciu.

CONTENTS

1. CHEN LIE ALGEBRAS

1.1. **The lower central series.** Let G be a group, and let $\Bbbk$ be a field of characteristic 0. There is then graded Lie algebra over $\Bbbk$,

$$\mathrm{gr}(G;\Bbbk) := \bigoplus_{k\geq 1}(\Gamma_k G/\Gamma_{k+1}G)\otimes_{\mathbb{Z}}\Bbbk, \tag{1}$$

where $\{\Gamma_k G\}_{k\geq 1}$ is the *lower central series* of the group G, which is defined inductively by the formulas: $\Gamma_1 G = G$, and $\Gamma_{k+1}G = [\Gamma_k G, G], k \geq 1$. The Lie bracket $[x,y]$ is induced from the group commutator $[x,y] = xyx^{-1}y^{-1}$. If $x \in \Gamma_r G$ and $y \in \Gamma_s G$, then the Lie bracket is given by $[x+\Gamma_{r+1}G, y+\Gamma_{s+1}G] = xyx^{-1}y^{-1} + \Gamma_{r+s+1}G$. If there is no confusion, we will omit the coefficients, and denote the associated graded $\Bbbk$-Lie algebra by $\mathrm{gr}(G)$.

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Example 1.1. The free group F with n generators. $\mathrm{gr}(F;\Bbbk) = \mathfrak{lie}$ the free Lie algebras with n generators.

Theorem 1.2 (Witt)**.** *The associated Lie algebra* $\mathrm{gr}(F)$ *is the free Lie algebra with basis* $\{x_1, \cdots, x_n\}$. *The number of basic commutators (LCS rank) of length k in the* F_k/F_{k+1} *is given by the necklace polynomial:*

$$N_k = \frac{1}{k} \sum_{d|k} \mu(d) \cdot n^{k/d},$$

where $\mu : \{1, 2, \cdots\} \to \{-1, 0, 1\}$ *is the Mobius function, defined by* $\mu(1) = 1$, $\mu(d) = 0$ *if d is divisible by a square, and* $\mu(p_1 \cdots p_m) = (-1)^m$ *if* $p_1, \cdots, p_m$ *are distinct prime numbers.*

1.2. **The Chen Lie algebra.** The *Chen Lie algebra* of a group G is defined to be the associated graded Lie algebra $\mathrm{gr}(G/G'';\Bbbk)$ of the derived quotient G/G'', where $G'' = [G', G']$ is the second derived subgroup. The quotient map $h\colon G \twoheadrightarrow G/G''$ induces surjective homomorphism $\mathrm{gr}(h;\Bbbk)\colon \mathrm{gr}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G/G'';\Bbbk)$. In particular, $\mathrm{gr}_k(h;\Bbbk)$ is isomorphic for $k \leq 3$.

The Chen ranks of G are defined as $\theta_k(G) := \mathrm{rank}(\mathrm{gr}_k(G/G'';\Bbbk))$. For a free group F of rank n, Chen ([1]) showed that

$$\theta_k(F) = (k-1)\binom{n+k-2}{k}, \tag{2}$$

for all $k \geq 2$. Let us also define the holonomy Chen ranks of G as $\bar{\theta}_k(G) = \dim(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(G;\Bbbk)$. It is readily seen that $\bar{\theta}_k(G) \geq \theta_k(G)$, with equality for $k \leq 2$.

Proposition 1.3. *Let* $G = F/\langle r \rangle$ *be a one-relator group, where* $F = \langle x_1, \ldots, x_n \rangle$, *and suppose G is* 1*-formal. Then*

$$\mathrm{Hilb}(\mathrm{gr}(G/G'', \Bbbk), t) = \begin{cases} 1 + nt - \dfrac{1 - nt + t^2}{(1-t)^n} & \text{if } r \in [F, F], \\ 1 + (n-1)t - \dfrac{1 - (n-1)t}{(1-t)^{n-1}} & \text{otherwise.} \end{cases}$$

2. Alexander invariants

2.1. **Fitting Ideals and Alexander polynomial.** Let R be a commutative ring with unit. Assume R is Noetherian and a unique factorization domain. Let M be a finitely-generated R-module. Then M admits a finite presentation

$$R^m \xrightarrow{A} R^n \to M \to 0.$$

The i-th *elementary ideal* (*Fitting ideas*) of M, denoted by $E_i(A) \subset R$, is the ideal in R generated by the $(n-i)\times(n-i)$ minors of the $n \times m$ matrix M. This ideal is independent of the presentation of A. Define $E_i(M) = R$ if $i \geq n$ and $E_i(M) = 0$ if $n - i > m$. Clearly, the Fitting ideals are increasing, $E_i(M) \subset E_{i+1}(A)$, for all $i \geq 0$. The ideal of maximal minors $E_0(A)$ is known as the *order ideal*.

2.2. **The Alexander module.** Let $\phi : G \twoheadrightarrow G_{\text{ab}}$ be a homomorphism onto the abelianization $G_{\text{ab}} = G/[G, G]$. Let $p\colon X^{\text{ab}} \to X$ be the corresponding maximal abelian cover. Denote $F = p^{-1}(x_0)$ the fiber of p over the basepoint. Then, $\pi_1(X^{\text{ab}}) \cong G' = [G, G]$ and $F \cong G_{\text{ab}}$ as a set. The exact sequence of the pair (X^{ab}, F) yields an exact sequence of $\mathbb{Z}[G_{\text{ab}}]$-modules,

$$0 \to H_1(X^{\text{ab}}; \mathbb{Z}) \to H_1(X^{\text{ab}}, F; \mathbb{Z}) \to H_0(F; \mathbb{Z}) \to H_0(X^{\text{ab}}; \mathbb{Z}) \to 0.$$

These are $\mathbb{Z}[G_{\text{ab}}]$-modules because G_{ab} acts on X^{ab} freely and transitively coming from deck transformations on (X^{ab}, F).

The $\mathbb{Z}[G_{\text{ab}}]$-modules $B(G) = H_1(X^{\text{ab}}; \mathbb{Z})$ and $A(G) = H_1(X^{\text{ab}}, F; \mathbb{Z})$ are called the *Alexander invariant* and *Alexander module* of X, relative to ϕ respectively.

Identify the kernel of $H_0(F; \mathbb{Z}) \to H_0(X^{\text{ab}}; \mathbb{Z})$ with the augmentation ideal, $I(G_{\text{ab}}) = \ker(\mathbb{Z}[G_{\text{ab}}] \xrightarrow{\epsilon} \mathbb{Z})$, we get the "Crowell exact sequence" of X,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X^{\text{ab}}; \mathbb{Z}) & \longrightarrow & H_1(X^{\text{ab}}, F; \mathbb{Z}) & \longrightarrow & \ker(\epsilon) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & B(G) & & A(G) & & I(G_{\text{ab}}) & & \end{array}$$

We can choose a 2 dimensional CW-complex to simply the computation, since we only need to compute the first homology.

$A(G) = \mathbb{Z}[G_{\text{ab}}] \otimes_{\mathbb{Z}[G]} I_G$ and $B(G) = G'/G''$, with $\mathbb{Z}[G_{\text{ab}}]$-module structure on $B(G)$ determined by the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

with G/G' acting on the cosets of G'' via conjugation: $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$.

2.3. **Massey's Theorem.** The module $B(G)$ has an I-adic filtration $\{I^k B(G)\}_{k \geq 0}$. Let $\text{gr}(B(G))$ be the associated graded module over the ring $\text{gr}(\Bbbk[G_{\text{ab}}])$, that is

$$\text{gr}(B(G)) = \bigoplus_{k \geq 0} I^k B(G)/I^{k+1} B(G).$$

In paper [5], Massey shows that $I^k B \cong \gamma_{k+2}(G/G'')$, and the following proposition.

Proposition 2.1. [5] *There exists an isomorphism*

$$\text{gr}_k(G/G'') \cong \text{gr}_{k-2}(B(G)),$$

for all $k \geq 2$. □

2.4. **The presentation.** Suppose G_{ab} is torsion-free with a basis $\{t_1, \cdots, t_n\}$, where $t_i = \alpha\varphi(x_i)$ with $\alpha : G \to G/G'$ and $\varphi : F_n \to G$ the canonical projections. So there is identification $\mathbb{Z}[H] \cong \Lambda = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ and the augmentation ideal I_G corresponds to the ideal $J = (t_1 - 1, \cdots, t_n - 1)$. Pick as basis elements, $t_i = \alpha\varphi(x_i)$, where $\varphi(x_i) : F_n \to G$ is the canonical projection.

Given a group G with a commutator-relators presentation $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$.

There is a presentation of I_G

$$(\mathbb{Z}[G])^m \xrightarrow{D_G=(\varphi\partial_i(r_j))} (\mathbb{Z}[G])^n \longrightarrow I_G \longrightarrow 0.$$

Tensor Λ to the above presentation, we get a finite presentation of Alexander module $A(G)$

$$\Lambda^m \xrightarrow{D_G=(\alpha\varphi\partial_i(r_j))} \Lambda^n \longrightarrow A(G) \longrightarrow 0, \tag{3}$$

where $\partial_i = \partial/\partial x_i : \mathbb{Z}F_n \to \mathbb{Z}F_n$ are the Fox free derivatives.

This $m \times n$ matrix $M = D_G = (\alpha\varphi\partial_i(r_j))$ is defined to be the *Alexander matrix* of G with entries in the Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$.

The *Alexander ideal* is the first elementary ideal of the Alexander module

$$E_1(A(G))$$

The i-th *Alexander polynomial* of M, denoted $\Delta_i(M)$, is a generator of the smallest principal ideal in R containing $E_i(M)$, that is, the greatest common divisor of all elements of $E_i(M)$. $\Delta_i(M)$ is well-defined only up to units in R.

The presentation for the Alexander invariant $B(G)$ is given by

$$\Lambda^{\binom{n}{3}+m} \xrightarrow{\Delta_G=\binom{\delta_3}{\Psi_2}} \Lambda^{\binom{n}{2}} \longrightarrow B(G) \longrightarrow 0, \tag{4}$$

where Ψ_2 is a map satisfying $\delta_2 \circ \Psi_2 = \alpha\varphi d_2$. Here, δ_2 is the second differential in Koszul complex and d_2 is given by Fox derivative.

2.5. **The link theory.** If $L = \widehat{\beta}$ is the closure of a pure braid $\beta \in P_n$, then the fundamental group has a presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$.

The homology groups of X are computed by Alexander duality,

$$\widetilde{H}_i(S^3 \setminus L) \cong \widetilde{H}^{3-i-1}(L),$$

that is $\widetilde{H}_0(X) = 0$, $H_1(X) = \mathbb{Z}^n$, $H_2(X) = \mathbb{Z}^{n-1}$, $H_{i\geq 3}(X) = 0$.

The chain complex of X^{ab} is given by

$$0 \longrightarrow C_3(X^{\text{ab}};\mathbb{Z}) \xrightarrow{\partial_3} C_2(X^{\text{ab}};\mathbb{Z}) \xrightarrow{\partial_2} C_1(X^{\text{ab}};\mathbb{Z}) \xrightarrow{\partial_1} C_0(X^{\text{ab}};\mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z}$$

$$0 \longrightarrow \Lambda^* \xrightarrow{\partial_3} \Lambda^m \xrightarrow{\partial_2} \Lambda^n \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z}$$

2.6. **The knot theory.**

Example 2.2 (Tame knot)**.** This theory comes from knot theory. Let K be a tame knot, and $X = S^3 \backslash K$ be the complement of K in S^3. Let $G = \pi_1(X)$ be the fundamental group. We already know that G is finitely generated $G_{\text{ab}} = \mathbb{Z}$ and $X \simeq K(G, 1)$. We have $\Lambda = \mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{Z}[[t]]$.

Then, we have $\tilde{H}_0(X^{\text{ab}}) = 0$, $H_0(X^{\text{ab}}) = \Lambda/(t-1)$, $H_i(X^{\text{ab}}) = 0$ for $i \geq 2$. The only significant one is $B(G) = H_1(X^{\text{ab}})$

3. Characteristics varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$). Without lose of generality, suppose X has only one single 0-cell x_0. Let $\Bbbk$ be a field and $\Bbbk^*$ be the group of units of $\Bbbk$. The cellular chain complex of X is denoted by $(C_i(X, \Bbbk), \partial_i)$. If the coefficient $\Bbbk = \mathbb{C}$, then denote the chain complex by $(C_i(X), \partial_i)$. If the universal cover of X is $\tilde{X} \to X$, then $C_i(\tilde{X})$ is a chain complex of module over $\mathbb{C}G$, where $G = \pi_1(X, x_0)$. The module structure

$$\mathbb{C}G \times C_i(\tilde{X}) \to C_i(\tilde{X})$$

is given by $g \cdot e_i$ and linear expansion. The group homomorphism $\mathrm{Hom}(G, \mathbb{C}^*)$ is an algebraic group, with multiplication $f_1 \circ f_2(g) = f_1(g) f_2(g)$ and identity $id(g) = 1$ for $g \in G$ and $f_i \in \mathrm{Hom}(G, \mathbb{C}^*)$. Since $\mathbb{C}^*$ is abelian group, we have

$$\mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^{b_i} \oplus \bigoplus_i \mathbb{Z}/k_i\mathbb{Z}, \mathbb{C}^*) = (\mathbb{C}^*)^{b_i} \oplus \bigoplus_i \mathbb{C}^*/k_i\mathbb{C}^*$$

Let $\rho : G \to \mathbb{C}^* \in \mathrm{Hom}(G, \mathbb{C}^*)$. The rank 1 local system on X is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$. There is a is a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \circ a$ for $a \in \mathbb{C}_\rho$ and $g \in G$.

Definition 3.1. The homology group of X with coefficient in $\mathbb{C}_\rho$ is defined by

$$H_i(X, \mathbb{C}_\rho) := H_i(C_*(\tilde{X}, \mathbb{C}) \otimes_{\mathbb{C}G} \mathbb{C}_\rho)$$

For $\rho = id \in \mathrm{Hom}(G, \mathbb{C}^*)$ gives the homology of X with $\mathbb{C}$ coefficient.

Definition 3.2. The *characteristic varieties* of X over $\mathbb{C}$ are the Zariski closed set

$$\mathcal{V}^i_d(X, \mathbb{C}) = \{\rho \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d\}$$

for $0 \leq i \leq k$ and $k > 0$.

Proposition 3.3.

(1) *There is a filtration*

$$\mathrm{Hom}(G, \mathbb{C}^*) \supseteq \mathcal{V}^i_1(X, \mathbb{C}) \supseteq \mathcal{V}^i_1(X, \mathbb{C}) \supseteq \cdots$$

(2) *The product rule for resonance varieties*

$$\mathcal{V}^i_1(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1) \times \mathcal{V}^q_1(X_2)$$

4. Resonance varieties

Let G be a group, and $A = H^*(G; \mathbb{C})$ be the cohomology algebra, with product operation given by the cup product of cohomology classes. For each $a \in A^1$, we have $a^2 = 0$.

Definition 4.1. The *Aomoto complex* of A is the cochain complex of finite-dimensional, complex vector spaces,

$$(A, a) : A^0 \xrightarrow{a \cup -} A^1 \xrightarrow{a \cup -} A^2 \xrightarrow{a \cup -} \cdots,$$

with differentials given by left-multiplication by a.

Definition 4.2. The *resonance varieties* of G are the sets

$$\mathscr{R}^i_d(G) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^i(A; a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

5. Some applications

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